THE APPROACH TO THE RIEMANN HYPOTHESIS VIA HILBERT–PÓLYA OPERATORS AND DENINGER COHOMOLOGY THEORIES

PRINTED BY TAYLOR DUPUY
AI GENERATED

Abstract. These notes formulate one Deninger cohomology theory with a flow, products, duality, and a compatible positive star. A realization whose spectral divisor is the completed zeta function would imply the Riemann hypothesis by Deninger's formal argument after Serre. We propose two conjectural comparisons with that theory: Flach–Morin Weil–Arakelov cohomology on eligible objects as its derived integer-twist fibres, and a multiplicity-preserving spectral probe of the trivial norm-one sector of CCM adelic cyclic homology as its degree-one part over number fields. Neither comparison nor the common positive theory is constructed here. Finite-field examples calibrate the axioms, while counterexamples show that integer-twist fibres alone miss noninteger eigenvalues, scalar trace spaces can lose zero multiplicities, and a naive arithmetic tensor square need not have discrete spectral blocks. The use of AI assistance is described in Section 14. This manuscript has not been checked by a human. Readers use it at their own risk and are responsible for independently verifying any claim, proof, computation, or citation they rely on.

Contents

1. Introduction and main results 1
2. Prior work 21
3. Conventions and analytic preliminaries 27
4. Finite-field calibration 31
5. Analytic and positivity conditions on Deninger structures 40
6. Purity and the conditional Riemann hypothesis 45
7. The class of zeta functions and the main theorem 50
8. The Künneth conjecture 78
9. Negative results 96
10. Candidate theories 113
11. Imported results 117
12. De Branges spaces and the Weil form 122
13. Open obligations 130
14. Software and AI assistance 131
Appendix A. Token accounting 145
References 146

1. Introduction and main results

1.1. **The problem and the comparison proposal.** The Riemann hypothesis asks whether every nontrivial zero of the Riemann zeta function $\zeta(s)$ has real part $1/2$. Its analogue for a

Date: 21 November 2026.

smooth projective curve over a finite field is a theorem of Weil. There the zeta function is determined by the action of Frobenius on cohomology, and the bound on its zeros follows from positivity for correspondences on the square of the curve. These notes ask whether a comparable cohomology and positivity statement can be formulated for $\operatorname{Spec}\mathbb{Z}$ and its arithmetic relatives.

The finite-field precedent separates two mathematical tasks. A cohomology theory expresses the zeta function through linear operators and supplies products and duality. A compatible positive form then controls the eigenvalues of those operators. In the higher-dimensional analogue, the Weil conjectures describe the expected zeta-function properties, while the standard conjectures propose geometric positivity sufficient for the cohomological argument. Deninger's program replaces Frobenius over a finite field by a real flow on a hypothetical arithmetic cohomology. The infinitesimal generator of that flow is to have the zeta zeros as eigenvalues, and a compatible positive form would force them onto the critical line.

The geometric input is part of the problem. The arithmetic curve $\operatorname{Spec}\mathbb{Z}$ has the primes as its closed points; its Arakelov compactification also includes the infinite place. An absolute or $\mathbb{F}_1$ geometry seeks a setting in which this curve has a useful square, playing the role of the surface in Weil's proof. Connes–Consani's arithmetic and scaling sites supply concrete characteristic-one constructions connected to this question: the arithmetic site's counting distribution gives the completed zeta function, and periodic orbits of the scaling site carry a Riemann–Roch theorem [CC14a, Theorems 2.6–2.7], [CC17, Theorem 5.17]. The global cohomology and intersection positivity needed to complete this analogy remain construction problems in the sources considered here.

A Deninger cohomology theory (Definition 1.21) is a functor from compactified schemes to Deninger structures: a graded flow with cup product and duality, whose positive star is a property, together with the zeta divisor as an axiom on the functor. Conjectures 1.26 and 1.28 ask for one such theory on the common arithmetic and finite-field input category, and their full versions add compact supports and Galois descent. Its positive form gives the conditional zero-line theorem, Theorem 6.3; for number fields this is Deninger's argument after Serre. The axiom list does not assert that such a global theory has been constructed.

The comparison proposal in Conjecture 10.2 gives two ways to test the same hypothetical theory. The derived fibres at integer twists should recover Flach–Morin cohomology on the objects where it is defined. Over a number field, a spectral probe of the trivial norm-one sector of CCM adelic cyclic homology should recover its degree-one zero spectrum with multiplicities. This extraction requires finite-dimensional generalized eigenspaces with discrete support; its topological and trace comparison remains open. The existing sources establish a finite-field fixed-twist comparison under a finite-generation hypothesis (Theorem 11.16) and an adelic spectral trace formula [CCM09, Theorem 4.16]; they do not construct these two comparisons to one positive arithmetic theory.

The manuscript first explains the cohomological precedents and the finite-field logarithmic suspension, then proves the formal implications and the negative tests. In particular, integer-twist fibres do not determine the full generator (Proposition 9.3), and scalar spectral models can lose zero multiplicities (Proposition 9.8). The proposed arithmetic-square Künneth and Hodge-index conditions require additional product data; their naive strong forms fail in the examples of Section 8. Sections 10 and 11 compare the candidate constructions with the single axiom list and state the exact source hypotheses.

1.2. **A common spectral formalism.** This subsection concerns the values of a cohomology theory: spectral cores with a flow, the Deninger axioms for a single datum, Deninger structures, and the purity implication. The source category of compactified schemes is fixed in Section 1.3.

Hermitian forms are linear in their first variable. For a nonzero meromorphic function Z on $\mathbb{C}$, write $\operatorname{div} Z = \sum_\rho \operatorname{ord}_\rho(Z)[\rho]$, with poles having negative order. The definitions of spectral core and positive infinitesimal polarization below isolate the linear algebra used in the main theorem.

Definition 1.1. Let V be a complex vector space and $\Theta : V \to V$ a linear map.

(a) For $\rho \in \mathbb{C}$ the *generalized eigenspace* $V_{\Theta\sim\rho}$ is $\bigcup_{k\geq 1} \ker(\Theta - \rho)^k$.
(b) (V, Θ) is a *spectral core* if V is the algebraic direct sum of the $V_{\Theta\sim\rho}$, $\rho \in \mathbb{C}$, each of them is finite-dimensional, and the set $\operatorname{spec}\Theta = \{\rho : V_{\Theta\sim\rho} \neq 0\}$ has no finite accumulation point.
(c) The *multiplicity* of ρ is $m(\rho) = \dim V_{\Theta\sim\rho}$.

The cohomology axioms use a flow in place of Frobenius and a unit adapted to the arithmetic base. The following definitions fix these objects before the axioms are stated.

Definition 1.2. Let V be a complex vector space which is the algebraic direct sum of finite-dimensional subspaces V_α, and let $\Theta : V \to V$ be a linear map preserving each V_α.

(a) The *flow* generated by Θ is the family of linear maps $\phi^t = e^{t\Theta}$, $t \in \mathbb{R}$, defined blockwise on each V_α; Θ is its *generator*.
(b) For $n \in \mathbb{Z}$ the *Tate twist* $V(n)$ is V with generator $\Theta - n$ (flow $e^{-nt}\phi^t$).
(c) $\mathbb{C}(-d)$ denotes the line $\mathbb{C}$ with generator multiplication by d.

Definition 1.3. Let $q > 1$ be real.

(a) The *arithmetic unit* is the spectral core $\mathbb{F}_1 = (\mathbb{C}, 0)$ with its multiplication.
(b) The *q-periodic unit* is the spectral core $\mathbb{F}_q = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}v^n$ (algebraic direct sum) with $\Theta v^n = (2\pi i n/\log q)v^n$ and multiplication $v^n v^m = v^{n+m}$.
(c) For a spectral core (U, Θ_U) and $k \in \mathbb{Z}$, $U(-k)$ is U with generator $\Theta_U + k$, as in Definition 1.2.

Over a number field the unit is $\mathbb{F}_1$; over a fixed finite field it is $\mathbb{F}_q$, whose periodic spectrum records all logarithms of Frobenius. The latter is Deninger's finite-Fourier-series model [Den98, pp. 167–168 and Proposition 3.1].

Definition 1.4. Let $d \geq 0$ be an integer.

(a) A *Weil datum of type* $(d, 0)$ is a tuple $(V^\bullet, \Theta, \cup, \mathrm{Tr}, *)$ consisting of spectral cores (V^j, Θ_j), $0 \leq j \leq 2d$, bilinear *cup products* $\cup : V^i \times V^k \to V^{i+k}$ (zero if $i + k > 2d$), a linear *trace* $\mathrm{Tr} : V^{2d} \to \mathbb{C}$, and antilinear *stars* $*_j : V^j \to V^{2d-j}$.
(b) Its *duality pairings* are $B_j(u, v) = \mathrm{Tr}(u \cup v)$ on $V^j \times V^{2d-j}$, and its *star forms* are $h_j(u, v) = \mathrm{Tr}(u \cup *_j v)$ on V^j.
(c) The *spectral divisor* of (V^j, Θ_j) is $\mathrm{Sp}_j = \sum_\rho m_j(\rho)[\rho]$, where m_j is the multiplicity function of Definition 1.1.

Definition 1.5 (Deninger axioms)**.** For an individual Weil datum D of type $(d, 0)$, a nonzero meromorphic function Z on $\mathbb{C}$, and a unit spectral core U with multiplication, we say that D *satisfies AWk for* (Z, U) when condition AWk below holds.

AW1 *(Grading, flow, and endpoints.)* The cup product is associative and graded-commutative with a unit $1 \in V^0$, and Θ is a derivation: $\Theta_{i+k}(u \cup v) = \Theta_i u \cup v + u \cup \Theta_k v$. There are isomorphisms of spectral cores $V^0 \cong U$, compatible with the multiplications and sending 1 to 1, and $V^{2d} \cong U(-d)$.

AW2 *(Spectral realization of the divisor.)* The divisor identity

$$\operatorname{div} Z = \sum_{j=0}^{2d} (-1)^{j+1} \operatorname{Sp}_j \tag{AW-div}$$

holds.

AW3 *(Duality.)* The trace is flow-equivariant into $\mathbb{C}(-d)$, that is, $\operatorname{Tr}(\Theta_{2d}x) = d\operatorname{Tr}(x)$, and for each j and ρ the pairing B_j restricts to a perfect pairing $V^j_{\Theta\sim\rho} \times V^{2d-j}_{\Theta\sim d-\rho} \to \mathbb{C}$.

AW4 *(Positive compatible star.)* For each j, $\Theta_{2d-j}{*_j} = {*_j}\Theta_j + (d-j){*_j}$, and h_j is a positive definite Hermitian form on V^j.

Conditions AW1–AW3 do not involve the stars, and we use them also for tuples $(V^\bullet, \Theta, \cup, \operatorname{Tr})$ without stars. These are conditions on a single datum. A cohomology theory assigns such data functorially to the objects of a source category, fixed in Section 1.3.

It is natural to expect the same internal algebra axioms to apply to a fixed positive-weight coefficient. The following obstruction shows why the coefficient theory requires different product targets.

Proposition 1.6 (Fixed-weight internal algebra obstruction)**.** *Let $d \geq 0$, $w \in \mathbb{R}$, and let $V^\bullet$ be a graded complex algebra of spectral cores in degrees $0, \dots, 2d$. Suppose Θ is a derivation, $\operatorname{Tr}(\Theta x) = (d+w)\operatorname{Tr}(x)$, and antilinear maps ${*_j} : V^j \to V^{2d-j}$ satisfy*

$$\Theta_{2d-j}{*_j} = {*_j}\Theta_j + (d-j){*_j}, \qquad h_j(u,v) = \operatorname{Tr}(u \cup {*_j}v).$$

If every h_j is positive definite Hermitian and $w \neq 0$, then every V^j is zero. If the algebra is nonzero and unital, its unit already forces $w = 0$.

Proof. The derivation, trace, and star laws give

$$h_j(\Theta u, v) + h_j(u, \Theta v) = (w+j)h_j(u,v).$$

On each finite invariant generalized eigenspace, $\Theta_j - (w+j)/2$ is skew-Hermitian and hence diagonalizable. Thus V^j is spanned by eigenvectors whose eigenvalues have real part $(w+j)/2$. By the derivation law, a nonzero product of eigenvectors in degrees i, k has eigenvalue equal to the sum of their eigenvalues. Its real part would be both $w+(i+k)/2$ and $(w+i+k)/2$, forcing $w = 0$. If $w \neq 0$, all products vanish, so all h_j vanish; positivity gives $V^j = 0$. In the unital case $\Theta 1 = 2\Theta 1$, hence $\Theta 1 = 0$. The displayed identity in degree zero gives $wh_0(1,1) = 0$. □

Deleting only the internal unit does not repair the positive-weight axioms. Coefficient products instead take values in tensor-product coefficients, and their pairings use the dual coefficient. It is also natural to expect that moving the weight between the trace and the star, or letting the unit carry it, avoids the obstruction; in fact neither does. With trace law $\operatorname{Tr}(\Theta x) = \tau \operatorname{Tr}(x)$ and star law $\Theta_{2d-j}{*_j} = {*_j}\Theta_j + \kappa_j{*_j}$, the computation in the proof gives $h_j(\Theta u, v) + h_j(u, \Theta v) = (\tau - \kappa_j)h_j(u,v)$, so only the difference $\tau - \kappa_j$ matters. Keeping the intended weight $w+j$ forces $\tau - \kappa_j = w + j$, and the unit argument applies unchanged. A derivation of a unital algebra kills its unit, so the unit cannot carry a nonzero weight without changing the derivation rule of AW1.

These are the core cohomological requirements used for the RH implication: algebra and flow, zeta realization, duality, and compatible positivity. The endpoint normalization is Deninger's [Den98, (3), p. 168]; the trace, duality, and star conditions follow his formulation and that of Flach–Morin [Den10, pp. 4–5], [FM20, Conjecture 2.3(iii)–(iv), p. 522]. Pullbacks are not required to preserve traces or stars, or to be isometries. Absolute squares need not satisfy these axioms and may have larger cohomology spaces (Theorem 8.12).

The additional Hilbert-space and trace-formula requirements AW5–AW6 are recorded separately in Definition 5.4. The regularized-determinant condition AW-det is stronger than the

divisor axiom AW2 (Definition 5.6). None is added to the hypotheses of the purity implication below. The next definitions isolate that implication's linear algebra and allow an arbitrary real weight offset; the internal Weil datum above is restricted to weight zero.

Definition 1.7. Let V be a complex vector space, $\Theta : V \to V$ a linear map, and $j \in \mathbb{R}$. A *positive infinitesimal polarization of weight* j on (V, Θ) is a positive definite Hermitian form h on V such that

$$h(\Theta u, v) + h(u, \Theta v) = j\, h(u, v) \qquad (u, v \in V). \tag{1.1}$$

Definition 1.8 (Graded spectral realization)**.** Let $d \geq 0$ be an integer, $w \in \mathbb{R}$, and $Z \not\equiv 0$ a meromorphic function on $\mathbb{C}$.

(a) A *graded spectral realization of type* (d, w) of Z consists of spectral cores (V^j, Θ_j), $0 \leq j \leq 2d$, with
$$\operatorname{div} Z = \sum_{j=0}^{2d} (-1)^{j+1} \sum_{\rho \in \mathbb{C}} \dim V^j_{\Theta \sim \rho}[\rho].$$

(b) Such a realization is *polarized* if each (V^j, Θ_j) carries a positive infinitesimal polarization of weight $w + j$.

The parameter w records the intended weights; it imposes no spectral restriction until a polarization is required. This definition keeps only the hypotheses needed for zero location. The datum of Definition 1.4 adds cup products, duality, and a star; its axioms AW1–AW4 imply a polarized realization of type $(d, 0)$ by Proposition 6.1. Weighted coefficients use this spectral definition directly, without an internal unit, cup product, or trace. A divisor identity is weaker than a regularized determinant identity and contains no normalization of the latter.

A Weil cohomology theory is a functor whose values carry the structure the standard conjectures speak about. The same organization applies here. The target category records the algebra, flow, and duality; the positive star is a property, as for polarizable Hodge structures; and the zeta realization is an axiom on the functor, because it refers to the prescribed function of the source object.

Definition 1.9 (Flow spaces)**.** Flow is the category whose objects are spectral cores (V, Θ) (Definition 1.1) and whose morphisms $f : (V, \Theta_V) \to (W, \Theta_W)$ are $\mathbb{C}$-linear maps with $f\Theta_V = \Theta_W f$.

For $n \in \mathbb{Z}$ the twist $V(n)$ has generator $\Theta - n$. The arithmetic unit $\mathbb{1}$ and the periodic units $\mathbb{1}_q$ of Definition 1.3 are objects of Flow with multiplications.

Proposition 1.10 (Exactness and additivity of divisors)**.** *Let $f : (V, \Theta_V) \to (W, \Theta_W)$ be a morphism in* Flow*, and let $\cdots \to V'^j \to V^j \to V''^j \to V'^{j+1} \to \cdots$ be a long exact sequence in* Flow *with $V'^j = V^j = V''^j = 0$ for $|j|$ large.*

(a) $\ker f$ *and* $\operatorname{coker} f$*, with the induced generators, are spectral cores, and for each $\rho \in \mathbb{C}$ the functor $V \mapsto V_{\Theta \sim \rho}$ is exact on* Flow*.*

(b) $\sum_j (-1)^j \operatorname{Sp}(V^j) = \sum_j (-1)^j \operatorname{Sp}(V'^j) + \sum_j (-1)^j \operatorname{Sp}(V''^j)$.

Proof of ((a)). A linear map commuting with the generators preserves generalized eigenspaces, so f is the direct sum of its restrictions $V_{\Theta \sim \rho} \to W_{\Theta \sim \rho}$. Kernels and cokernels are computed blockwise. Each block is finite-dimensional, and the spectrum of $\ker f$ or $\operatorname{coker} f$ lies in that of V or W, so it has no finite accumulation point. □

Proof of ((b)). By ((a)) the restriction of the sequence to the ρ-blocks is a bounded exact sequence of finite-dimensional spaces, so the alternating sum of its dimensions vanishes. Comparing coefficients of $[\rho]$ gives the identity. □

It is natural to expect Flow to be symmetric monoidal under $V \otimes W$ with generator $\Theta \otimes 1 + 1 \otimes \Theta$, as the target of a Weil cohomology theory is. In fact the algebraic tensor product of two spectral cores need not be a spectral core (Theorem 8.12), so no tensor product is part of the structure. The relative tensor product over $\mathbb{F}_q$ is the correct product on the finite-field objects (Proposition 8.17 and Theorem 8.23).

Definition 1.11 (Deninger structures)**.** Let $d \geq 0$ be an integer and let U be $\mathbb{F}$ or some $\mathbb{F}_q$.

(a) A *Deninger structure of dimension d with unit U* is a tuple $D = (V^\bullet, \Theta, \cup, \mathrm{Tr})$ of spectral cores (V^j, Θ_j) for $0 \leq j \leq 2d$, bilinear products $\cup : V^i \times V^k \to V^{i+k}$, zero if $i + k > 2d$, and a linear trace $\mathrm{Tr} : V^{2d} \to \mathbb{C}$, satisfying AW1 and AW3 for U.

(b) A *morphism* of Deninger structures is a graded unital algebra homomorphism commuting with the generators.

(c) $\mathrm{Den}_d(U)$ is the category of Deninger structures of dimension d with unit U. The category $\mathrm{Den}^\natural$ has all Deninger structures as objects and all morphisms of (b), including maps between different pairs (d, U). Thus $\mathrm{Den}_d(U)$ is a full subcategory of $\mathrm{Den}^\natural$.

Explicitly, the product is associative, graded-commutative, and unital, Θ is a derivation, $V^0 \cong U$ compatibly with the multiplications and units, $V^{2d} \cong U(-d)$, the trace is flow-equivariant into $\mathbb{C}(-d)$, and the duality pairings are perfect on generalized eigenspaces. The divisor condition AW2 is omitted because it refers to an external function, and the star of AW4 because its existence is a property. Morphisms need not preserve traces.

Proposition 1.12 (Disjoint-union obstruction)**.** *Let $\mathcal{C}$ be a category with an assignment of a pair (d_X, U_X) to each object X. If $\mathcal{C}$ has a morphism $f : X \to Y$ with $(d_X, U_X) \neq (d_Y, U_Y)$, there is no functor $H : \mathcal{C}^{\mathrm{op}} \to \coprod_{d,U} \mathrm{Den}_d(U)$ for which $H(Z) \in \mathrm{Den}_{d_Z}(U_Z)$ for every object Z.*

Proof. The image of f would be a morphism $H(Y) \to H(X)$ between distinct components of the disjoint union, but that category has no such morphism. □

Definition 1.13 (Polarizable Deninger structures)**.** Let $D = (V^\bullet, \Theta, \cup, \mathrm{Tr})$ be a Deninger structure of dimension d with unit U.

(a) A *polarization* of D is a family of antilinear maps $*_j : V^j \to V^{2d-j}$ such that the Weil datum $(V^\bullet, \Theta, \cup, \mathrm{Tr}, *)$ satisfies AW4.

(b) D is *polarizable* if it has a polarization.

(c) $\mathrm{Den}_d^{\mathrm{pol}}(U)$ is the full subcategory of $\mathrm{Den}_d(U)$ of polarizable Deninger structures.

A polarized Deninger structure is a Weil datum satisfying AW1, AW3, and AW4. When it also satisfies AW2 for a function Z, it is a polarized graded spectral realization of Z of type $(d, 0)$ (Proposition 6.1), and Theorem 1.15 below applies.

It is natural to expect morphisms of polarizable structures to preserve the forms h_j. In fact, under the compatibility of traces with cycle classes expected in [Den25, (2)–(5), pp. 4–5], the pullback along the map of Arakelov compactifications induced by a finite extension L/K of number fields multiplies the degree-one form by $[L : K]$ [Den25, (8), p. 5]. Polarizability is therefore a property of objects, as for polarizable Hodge structures, and $\mathrm{Den}_d^{\mathrm{pol}}(U)$ is a full subcategory.

Definition 1.14 (Deninger modules)**.** Let $D = (V^\bullet, \Theta, \cup, \mathrm{Tr})$ be a Deninger structure.

(a) A *Deninger module over D* is a family $(M^j, \Theta_j)_{j \in \mathbb{Z}}$ of spectral cores, zero for $|j|$ large, with an associative graded action $V^i \times M^k \to M^{i+k}$ of the algebra $V^\bullet$ for which Θ is a derivation.

(b) Mod_D is the category of Deninger modules over D, with graded $V^\bullet$-linear maps commuting with the generators as morphisms.

Deninger modules carry compactly supported cohomology of non-proper objects and coefficient cohomology of weight $w \neq 0$. No unit, trace, or duality is required. A fixed positive-weight coefficient cannot carry the unital structure of Definition 1.11 (Proposition 1.6); its products and dualities take values in tensor-product and dual coefficients, as in Remark 7.31.

Theorem 1.15 (General cohomological purity implication)**.** *Let $d \geq 0$ be an integer, $w \in \mathbb{R}$, and $Z \not\equiv 0$ meromorphic on $\mathbb{C}$. Suppose that Z has a polarized graded spectral realization of type (d, w). Then each Θ_j is diagonalizable, every one of its eigenvalues has real part $(w+j)/2$, and*

$$\operatorname{ord}_\rho Z = \begin{cases} (-1)^{j+1} \dim \ker(\Theta_j - \rho), & 2\operatorname{Re}\rho - w = j \in \{0, \dots, 2d\}, \\ 0, & 2\operatorname{Re}\rho - w \notin \{0, \dots, 2d\}. \end{cases}$$

Thus zeros occur only on the odd-degree lines, poles only on the even-degree lines, and the divisor of Z determines each degreewise spectral divisor without cancellation between degrees.

Proof. If $\Theta_j v = \rho v$ and $v \neq 0$, the polarization identity gives $(\rho + \bar{\rho} - w - j)h_j(v, v) = 0$. Positivity therefore gives $\operatorname{Re}\rho = (w + j)/2$. On every finite-dimensional invariant subspace, $\Theta_j - (w + j)/2$ is skew-Hermitian and hence diagonalizable. Applying this to every generalized eigenspace proves diagonalizability on the algebraic direct sum. The lines $(w+j)/2$ are distinct, so at each point the divisor identity has at most one nonzero summand. □

The theorem concerns a single function and its realization; its proof uses no source geometry. This is Deninger's infinitesimal version of Serre's argument [Den94, (7.11)–(7.13), pp. 32–33], stated for arbitrary real weight offset and without unnecessary geometric hypotheses. Theorem 7.28 gives its cup-product and star formulation for $w = 0$. Higher-dimensional arithmetic zeta functions are not expected to have just one critical line: Proposition 7.27 proves that already $\zeta_{\mathbb{P}^1_{\mathbb{Z}}}(s) = \zeta(s)\zeta(s-1)$ has zeros in two disjoint strips.

Definition 1.16. Let ζ be the Riemann zeta function and let K be a number field with r_1 real places, r_2 complex places, discriminant D_K, and Dedekind zeta function ζ_K.

(a) The *archimedean gamma factors* are $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$.
(b) The *completed zeta function* is $\Lambda(s) = \Gamma_{\mathbb{R}}(s)\zeta(s)$.
(c) The *Riemann ξ-function* is $\xi(s) = \frac{1}{2}s(s-1)\Lambda(s)$.
(d) The *Riemann Ξ-function* is $F(z) = \Xi(z) = \xi(\frac{1}{2} - iz)$.
(e) The *completed Dedekind zeta function* is $\Lambda_K(s) = |D_K|^{s/2}\Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}\zeta_K(s)$.
(f) The *Dedekind ξ-function* is $\xi_K(s) = s(s-1)\Lambda_K(s)$, and its zeros are the *nontrivial zeros* of ζ_K.

Proposition 1.17 (Positive realization of the zeta zeros)**.** *Let ξ be the completed Riemann zeta function of Definition 1.16, and let (V, Θ) be a spectral core. Suppose that, for every $\rho \in \mathbb{C}$,*

$$\dim V_{\Theta \sim \rho} = \operatorname{ord}_\rho \xi,$$

and let h be a positive infinitesimal polarization of weight one on (V, Θ) (Definition 1.7). Then RH holds.

Proof. For each zero ρ of ξ, the multiplicity hypothesis gives a nonzero finite-dimensional generalized ρ-eigenspace, which contains an eigenvector $v \neq 0$. Substituting $u = v$ into (1.1) yields $(2\operatorname{Re}\rho - 1)h(v, v) = 0$. Since $h(v, v) > 0$, one has $\operatorname{Re}\rho = \frac{1}{2}$. □

The multiplicity hypothesis is the exact zero-spectrum requirement; positivity alone does not identify any eigenvalue with a zeta zero. The form must satisfy (1.1) for the same generator Θ; an arbitrary positive inner product is insufficient (Proposition 7.7(ii)).

1.3. **Geometric instances and the uniform conjectures.**

Definition 1.18 (Arakelov compactification)**.** Let X be a regular projective flat scheme over $\operatorname{Spec}\mathbb{Z}$. Its *Arakelov compactification* is the arithmetic pair $\overline{X} = (X, X_\infty)$, where X_∞ consists of $X(\mathbb{C})$ with its complex-conjugation action F_∞ and a Kähler form ω satisfying $F_\infty^*\omega = -\omega$. The dimension of $\overline{X}$ is the Krull dimension of X.

This is the convention of [FM20, Notation 2.2, pp. 520–522]. We use morphisms induced by morphisms of the underlying arithmetic schemes and their conjugation-equivariant maps at infinity; these morphisms are not required to preserve the chosen metrics. In particular, for a number field K, the object $\overline{X}_K = (\operatorname{Spec}\mathcal{O}_K, X_{K,\infty})$ has dimension one. The pair is not an ordinary scheme, and no product over an absolute base is included in this convention. The broader framework of [FM20, Conjecture 2.3, p. 522] also considers separated finite-type arithmetic schemes with compact supports. Positivity below concerns the projective regular objects.

Definition 1.19 (Compactified schemes)**.**

(a) A *compactified scheme* $\overline{X}$ is a pair (X, X_∞), where X is a separated scheme of finite type over $\operatorname{Spec}\mathbb{Z}$ and X_∞ is the set $X(\mathbb{C})$ with its complex conjugation F_∞, together with, when X is projective, flat, and regular over $\operatorname{Spec}\mathbb{Z}$, a Kähler form ω on $X(\mathbb{C})$ with $F_\infty^*\omega = -\omega$.

(b) $\overline{\mathrm{Sch}}$ is the category of compactified schemes. A morphism $\overline{X} \to \overline{Y}$ is a morphism of schemes $X \to Y$ together with the induced F_∞-equivariant map $X(\mathbb{C}) \to Y(\mathbb{C})$; the Kähler forms are not required to be preserved.

(c) $\overline{\mathrm{Sch}}^{\mathrm{reg}}$ is the full subcategory of objects with X regular of pure dimension.

(d) $\overline{\mathrm{Sch}}^{\mathrm{proj}}$ is the full subcategory of objects with X projective, flat, and regular over $\operatorname{Spec}\mathbb{Z}$, or with X smooth and projective over a finite field.

(e) $\overline{\mathrm{Sch}}_1$ is the full subcategory of $\overline{\mathrm{Sch}}^{\mathrm{proj}}$ whose objects are the finite disjoint unions of the $\overline{X}_K = (\operatorname{Spec}\mathcal{O}_K, X_{K,\infty})$ for number fields K.

(f) For a prime power q, $\mathrm{SmProj}_{\mathbb{F}_q}$ is the full subcategory of $\overline{\mathrm{Sch}}^{\mathrm{proj}}$ of smooth projective varieties over $\mathbb{F}_q$, and $\mathrm{SmProj}_{\mathbb{F}}$ is the full subcategory of smooth projective varieties over finite fields.

(g) For $\overline{X} \in \overline{\mathrm{Sch}}^{\mathrm{proj}}$, put $d_X = \dim X$ and $w_X = 0$. The *prescribed function* Z_X is the completed zeta function $\zeta(X,s)\zeta(X_\infty,s)$ of [FM20, pp. 520–521] if X is flat over $\operatorname{Spec}\mathbb{Z}$, and $\zeta_X(s)$ if X is a variety over a finite field.

(h) For connected $\overline{X} \in \overline{\mathrm{Sch}}^{\mathrm{proj}}$, the *unit* U_X is $\nVdash$ if X is flat over $\operatorname{Spec}\mathbb{Z}$, and $\nVdash_q$ if X is a variety over a finite field with $H^0(X, \mathcal{O}_X) \cong \mathbb{F}_q$.

For a variety over a finite field $X(\mathbb{C})$ is empty, so its compactified scheme is $(X, \varnothing)$. The objects $\overline{X}_K$ are finite flat over $\operatorname{Spec}\mathbb{Z}$; curves over finite fields are not objects of $\overline{\mathrm{Sch}}_1$. The bar in $\overline{\mathrm{Sch}}$ indicates the compactification, as it does in $\overline{X}$, while Sch denotes the category of schemes. The connected objects of $\overline{\mathrm{Sch}}^{\mathrm{proj}}$ are the projective regular class of [FM20, Notation 2.2, pp. 521–522], with the morphisms made explicit, together with the smooth projective varieties over finite fields.

Remark 1.20 (Prescribed data and ambient geometries)*.* The data d_X, w_X, Z_X, and U_X of Definition 1.19 are attached to the objects of $\overline{\mathrm{Sch}}^{\mathrm{proj}}$. The meromorphic continuation of every Z_X is part of Conjecture 1.26 below. The weight offset is zero for every object, as Proposition 1.6 forces for unital data; coefficients of nonzero weight use graded spectral realizations (Definition 1.8) and Deninger modules (Definition 1.14). A proposed geometry $\mathcal{G}$ over an absolute point enters through a comparison functor $\overline{\mathrm{Sch}} \to \mathcal{G}$ together with its cohomological comparisons. Agreement of zeta functions alone is not such a comparison: a degreewise comparison

must intertwine the generators and any pairings it transports. Products in $\mathcal{G}$ need not carry Deninger structures, since the tensor square of the Weil datum of an arithmetic curve is not a spectral core (Theorem 8.12). In particular a Connes–Consani model for $\operatorname{Spec}\mathbb{Z}$ has the same target function Λ as the Arakelov model; the arithmetic and scaling sites are semiringed topoi, not alternative definitions of an Arakelov pair [CC15, §1, pp. 1–2].

Definition 1.21 (Deninger cohomology theory)**.** For connected $\overline{X} \in \overline{\mathrm{Sch}}^{\mathrm{proj}}$ let d_X, Z_X, and U_X be as in Definition 1.19.

(a) A *Deninger cohomology theory* is a functor $H : (\overline{\mathrm{Sch}}^{\mathrm{proj}})^{\mathrm{op}} \to \mathrm{Den}^{\natural}$ with $H(\overline{X}) \in \mathrm{Den}_{d_X}(U_X)$ for every connected $\overline{X}$, satisfying F1 and F2 below.
(b) A *full Deninger cohomology theory* is a Deninger cohomology theory H together with an extension H_c to $\overline{\mathrm{Sch}}$, contravariant for proper morphisms and covariant for open immersions, whose value on $\overline{X}$ is a Deninger module over $H(\overline{X}_{\mathbb{Q}})$ or over $H(\operatorname{Spec}\mathbb{F}_q)$, satisfying F3 and F4 below.
(c) A Deninger cohomology theory, full or not, is *polarized* if $H(\overline{X})$ is polarizable (Definition 1.13) for every connected $\overline{X} \in \overline{\mathrm{Sch}}^{\mathrm{proj}}$.

The axioms are the following.

F1 *(Endpoints.)* For connected $\overline{X}$, the isomorphisms $H^0(\overline{X}) \cong U_X$ and $H^{2d_X}(\overline{X}) \cong U_X(-d_X)$ of Definition 1.11 hold.
F2 *(Zeta realization.)* For connected $\overline{X} \in \overline{\mathrm{Sch}}^{\mathrm{proj}}$, $\operatorname{div} Z_X = \sum_j (-1)^{j+1} \operatorname{Sp}(H^j(\overline{X}))$.
F3 *(Localization.)* For a closed subscheme $Y \subset X$ with open complement W, there is a long exact sequence $\cdots \to H^j_c(W) \to H^j_c(X) \to H^j_c(Y) \to H^{j+1}_c(W) \to \cdots$ in Flow.
F4 *(Galois descent.)* For a finite Galois extension L/K of number fields with group G, pullback along $\overline{X}_L \to \overline{X}_K$ induces $H^\bullet(\overline{X}_K) \cong H^\bullet(\overline{X}_L)^G$ compatibly with the generators and products.

The structure map $\mathbb{P}^1_{\mathbb{F}_q} \to \operatorname{Spec}\mathbb{F}_q$ is a morphism in $\overline{\mathrm{Sch}}^{\mathrm{proj}}$ between connected objects of dimensions 1 and 0. Proposition 1.12 therefore rules out the disjoint union of the $\mathrm{Den}_d(U)$ as the target of such a theory. The cross-component morphisms in $\mathrm{Den}^{\natural}$ remove this definitional obstruction.

A full Deninger cohomology theory is in particular a Deninger cohomology theory. Axiom F1 restates the endpoint clause of AW1, and F2 is AW2 for (Z_X, U_X). Hence the value $H(\overline{X})$ on a connected object is a datum without stars satisfying AW1–AW3 for (Z_X, U_X), pullbacks preserve cup products and commute with the generators, and a polarization of $H(\overline{X})$ supplies stars satisfying AW4. By Theorem 1.15, a polarized theory places the zeros of each odd-degree determinant on $\operatorname{Re} s = j/2$; this is [FM20, Conjecture 2.3(v), p. 522]. Axiom F4 is the descent condition of Definition 7.10. The compactly supported version of F2 requires archimedean factors for non-proper objects, which are not specified here.

Proposition 1.22 (Relative structure)**.** *Let H be a Deninger cohomology theory, K a number field, and $\pi : \overline{X} \to \overline{X}_K$ a morphism in $\overline{\mathrm{Sch}}^{\mathrm{proj}}$ with $\overline{X}$ connected. Then $H(\pi)$ makes $H(\overline{X})$ a graded $H(\overline{X}_K)$-algebra, and Θ is a derivation for this action.*

Proof. By functoriality $H(\pi) : H(\overline{X}_K) \to H(\overline{X})$ is a morphism in $\mathrm{Den}^{\natural}$, a graded unital algebra homomorphism commuting with the generators. Composing it with the cup product of $H(\overline{X})$ gives the action, and Θ is a derivation of that cup product by AW1. □

The proposition records what "relative over $\mathcal{O}_K$" means for a theory on $\overline{\mathrm{Sch}}^{\mathrm{proj}}$; it is a consequence of functoriality and not an additional axiom.

Proposition 1.23 (Finite-field calibration)**.** *Let H be a Deninger cohomology theory, and let X be a smooth projective geometrically connected variety of dimension d over $\mathbb{F}_q$. Let $\ell \nmid q$ be a prime, fix a field isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$, let $H^j_\ell(X) = H^j(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) \otimes \mathbb{C}$ with geometric Frobenius* F*, and let $(\widehat{H}^j_\ell(X), \Theta_j)$ be its logarithmic suspension: the algebraic direct sum, over all $\rho \in \mathbb{C}$ with q^ρ an eigenvalue of* F*, of copies of the generalized eigenspace $H^j_\ell(X)_{(q^\rho)}$, on which Θ_j is $\rho + N/\log q$ with $N = \log(q^{-\rho}\mathrm{F})$ nilpotent (Definition 4.15).*

(i) If $d = 1$, then $\mathrm{Sp}(H^1(X)) = \mathrm{Sp}(\widehat{H}^1_\ell(X))$.

(ii) If H is polarized, then for every j, $\mathrm{Sp}(H^j(X)) = \mathrm{Sp}(\widehat{H}^j_\ell(X))$, the generator of $H^j(X)$ is diagonalizable, and $H^j(X) \cong \widehat{H}^j_\ell(X)$ in Flow *if and only if* F *is semisimple on $H^j_\ell(X)$.*

Proof of ((i)). By Theorem 7.24(ii), $\operatorname{div}\zeta_X = \sum_j (-1)^{j+1}\,\mathrm{Sp}(\widehat{H}^j_\ell(X))$. Here $U_X = \mathbb{F}_q$. The suspension of $(H^0_\ell(X), \mathrm{F}) = (\mathbb{C}, 1)$ is $\mathbb{F}_q$, and that of $(H^2_\ell(X), \mathrm{F}) = (\mathbb{C}, q)$ has eigenvalues $1 + 2\pi i n/\log q$, $n \in \mathbb{Z}$, each simple, so it is $\mathbb{F}_q(-1)$. By F1, $H^0(X) \cong \mathbb{F}_q$ and $H^2(X) \cong \mathbb{F}_q(-1)$, so the degree-zero and degree-two terms agree on the two sides, and F2 gives $\mathrm{Sp}(H^1(X)) = \mathrm{Sp}(\widehat{H}^1_\ell(X))$. □

Proof of ((ii)). A polarization of $H(X)$ and F2 make $(H^j(X), \Theta_j)_j$ a polarized graded spectral realization of ζ_X of type $(d, 0)$ (Proposition 6.1). By Theorem 1.15 each generator is diagonalizable, and $\dim\ker(\Theta_j - \rho \mid H^j(X)) = (-1)^{j+1}\operatorname{ord}_\rho \zeta_X$ when $2\operatorname{Re}\rho = j$, while $H^j(X)$ has no other eigenvalues. On the other side, $\widehat{H}^j_\ell(X)_{\Theta\sim\rho}$ is the generalized q^ρ-eigenspace of F on $H^j_\ell(X)$. By Deligne's theorem [Del74, Théorème (1.6), p. 276] every eigenvalue α of F on $H^j_\ell(X)$ has $|\alpha| = q^{j/2}$, so $\widehat{H}^j_\ell(X)$ has spectrum on $\operatorname{Re}\rho = j/2$, and in the identity of Theorem 7.24(ii) only the degree $j = 2\operatorname{Re}\rho$ contributes at ρ. Hence $\dim \widehat{H}^j_\ell(X)_{\Theta\sim\rho} = (-1)^{j+1}\operatorname{ord}_\rho\zeta_X = \dim H^j(X)_{\Theta\sim\rho}$ for all j and ρ. Two spectral cores with diagonalizable generators and the same multiplicities are isomorphic in Flow. The generator of $\widehat{H}^j_\ell(X)$ is diagonalizable if and only if every nilpotent part N vanishes, that is, if and only if F is semisimple on $H^j_\ell(X)$. □

It is natural to expect the comparison with ℓ-adic cohomology to be an axiom, as in a Weil cohomology theory. In fact the spectral part of it follows from F1 and F2 for curves, and from polarization in every dimension, by Proposition 1.23. The isomorphism there is obtained degree by degree from multiplicities; it is neither natural in X nor compatible with cup products, and it is an isomorphism with $\widehat{H}^j_\ell(X)$ only when Frobenius is semisimple, which is not known in general. A natural comparison compatible with products remains a separate requirement, stated in Conjecture 1.26 below.

Proposition 1.24 (The divisor does not determine degreewise spectra)**.** *Let $d \geq 2$, let U be $\mathbb{F}$ or some $\mathbb{F}_q$, let $D = (V^\bullet, \Theta, \cup, \mathrm{Tr}) \in \mathrm{Den}_d(U)$, and let $\rho \in \mathbb{C}$ and $1 \leq i \leq 2d-2$. There is $D' = (V'^\bullet, \Theta, \cup, \mathrm{Tr})$ in $\mathrm{Den}_d(U)$ with $\sum_j (-1)^{j+1}\,\mathrm{Sp}(V'^j) = \sum_j (-1)^{j+1}\,\mathrm{Sp}(V^j)$ and $\mathrm{Sp}(V'^i) = \mathrm{Sp}(V^i) + \mathrm{Sp}(W)$, where W is U with generator $\Theta_U + \rho$.*

Proof. Let W' be U with generator $\Theta_U + d - \rho$, and write $w_u \in W$, $w'_u \in W'$ for the element $u \in U$. Fix the isomorphism $\phi : U \cong V^0$ of AW1. The pairing B_0 restricted to $V^0_{\Theta\sim 0} \times V^{2d}_{\Theta\sim d}$ is perfect and $V^0_{\Theta\sim 0} = \mathbb{C}\cdot 1$, so there is $t \in V^{2d}$ with $\Theta t = dt$ and $\mathrm{Tr}(t) = 1$. For $k \in \{i, i+1\}$ add a copy W_k of W in degree k and a copy W'_k of W' in degree $2d-k$; these degrees lie in $[1, 2d-1]$. Let V^0 act on W_k and W'_k through ϕ^{-1} and the multiplication of U, set $w_u \cup w'_{u'} = \phi(uu')\cup t$ for $w_u \in W_k$, $w'_{u'} \in W'_k$, with the graded-commutative sign for the opposite order, and let all other products involving the added summands vanish. The trace is unchanged. The derivation rule holds because the shifts ρ and $d-\rho$ add to the eigenvalue d of t, and associativity holds because

every product of an added element with an element of positive degree vanishes or lands in V^{2d}, whose products with elements of positive degree vanish. On generalized eigenspaces, $W_\sigma = U_{\sigma-\rho}$ pairs with $W'_{d-\sigma} = U_{\rho-\sigma}$ by $(u, u') \mapsto \mathrm{Tr}(\phi(uu') \cup t)$, which is perfect because the multiplication pairs U_α perfectly with $U_{-\alpha}$ into $U_0 = \mathbb{C} \cdot 1$. The endpoints are unchanged. The added divisor is $(-1)^{i+1} \mathrm{Sp}(W) + (-1)^{i+2} \mathrm{Sp}(W) + (-1)^{2d-i+1} \mathrm{Sp}(W') + (-1)^{2d-i} \mathrm{Sp}(W') = 0$. □

It is natural to expect F2 to determine the spectrum of each $H^j(X)$ for a variety X over $\mathbb{F}_q$, as it does for curves. In fact for $d \geq 2$ it determines only the alternating sum: for a surface, Proposition 1.24 with $i = 1$ adds the same spectrum to degrees one and two, and the same dual spectrum to degrees two and three, without changing the divisor. The structure D' is not polarizable, since a polarization would place the spectrum of W on two different lines (Theorem 1.15); this is why part ((ii)) of Proposition 1.23 needs polarization.

Remark 1.25 (Terminology and scope). *Deninger cohomology theory* means Definition 1.21, and the *Deninger axioms* AW1–AW4 of Definition 1.5 are conditions on a single datum. This is our terminology, not a claim of existence or uniqueness. Flach–Morin denote Deninger's conjectural groups by $H^\bullet_{\mathrm{dyn},c}$ and $H^\bullet_{\mathrm{dyn}}$; their separately constructed fixed-twist groups are called *Weil–Arakelov cohomology* [FM20, abstract, Conjecture 2.3, and §3]. We call these fixed-twist groups *Flach–Morin cohomology* in these notes. This is a local shorthand for their Weil–Arakelov cohomology, not the name used in the cited paper. Their proposed comparison [FM20, Conjecture 4.1, p. 527] makes them candidate fixed-twist data for a Deninger cohomology theory; it does not make them an instance of Definition 1.21. The divisor condition F2 determines alternating spectral multiplicities; it does not specify the functorial maps, cup products, or positive stars. For a dimension-one object with prescribed endpoints, it does fix the degree-one spectral divisor. This is a spectral constraint, not a uniqueness theorem for the cohomology theory.

The common substitutions in Theorem 1.15 are as follows. The table names the required realization, not its existence in every row.

Input	(d_X, w_X) and Z_X	Realization and consequence
Number field K	$(1, 0)$; Λ_K	Arakelov or a specified absolute model; zeros on $\mathrm{Re}\, s = 1/2$.
Smooth projective $X/\mathbb{F}_q$	$(\dim X, 0)$; ζ_X	Logarithmic suspension of Frobenius; degree-j line $\mathrm{Re}\, s = j/2$.
Regular projective flat $X/\mathbb{Z}$	$(\dim X, 0)$; completed ζ_X	Conjectural global realization; odd-degree zero lines and even-degree pole lines.
Pure weight-w coefficients over a number field	$(1, w)$; completed $L(M, s)$	Requires a polarized coefficient spectral realization; middle line $\mathrm{Re}\, s = (w+1)/2$.

For finite fields the suspension uses *all* logarithms of each Frobenius eigenvalue: $q^\rho = \alpha$ and $\mathrm{Re}\, \rho = \log|\alpha| / \log q$. The unit is the periodic object $\mathbb{1}_q$ of Definition 1.3, rather than the arithmetic unit $(\mathbb{C}, 0)$ (Proposition 7.23). Theorem 7.24 supplies the structural comparison; full positive polarization also requires Frobenius semisimplicity. A global function field is the curve case of that row. The coefficient row is conditional on a specified coefficient category and analytic continuation; it does not extend the number-field theorem to all automorphic L-functions. The arithmetic squares of Section 8 require the larger flow-module formalism and are not additional spectral-core objects in this table.

The following conjectures ask for a Deninger cohomology theory (Definition 1.21), in a core and a full version, together with a finite-field comparison. Existence, zeta identification, and duality are separated from positivity. A realization through another source geometry must supply an input functor and these same cohomological structures.

Conjecture 1.26 (Uniform cohomological realization)**.** *The prescribed functions Z_X on $\overline{\mathrm{Sch}}^{\mathrm{proj}}$ admit meromorphic continuation to $\mathbb{C}$, and there is a Deninger cohomology theory H whose restriction to each $\mathrm{SmProj}_{\mathbb{F}_q}$ is naturally isomorphic, with generators and products, to the logarithmic suspension of a chosen complex realization of ℓ-adic cohomology, $\ell \nmid q$.*

Conjecture 1.27 (Full uniform cohomological realization)**.** *The statement of Conjecture 1.26 holds with H a full Deninger cohomology theory.*

For $\overline{X} = \overline{X}_K$ the endpoint data are $(H^0, \Theta_0) = (\mathbb{C}, 0)$ and $(H^2, \Theta_2) = (\mathbb{C}, 1)$, by F1 with $U_X = \nVdash$. The natural comparison with ℓ-adic cohomology is part of the conjecture rather than of Definition 1.21; its spectral content is a consequence of the axioms in the cases of Proposition 1.23.

The cohomology, trace formula, duality, and determinant expectations underlying Conjecture 1.26 are [Den10, Conjecture 2, p. 3] and [FM20, Conjecture 2.3(i)–(iii), p. 522]. The completed function includes Serre's archimedean factors of the generic fibre, as in Section 7.6; multiplication by a nowhere-vanishing entire function leaves its divisor unchanged. Definition 1.11 records the cup-product and duality requirements; Section 5 develops them and the stronger regularized-determinant condition. An analytic construction of those determinants, identified with the Euler and gamma factors on their convergence half-plane, would also supply the meromorphic continuation asserted here.

Conjecture 1.28 (Uniform positivity)**.** *There is a Deninger cohomology theory H as in Conjecture 1.26 that is polarized.*

Conjecture 1.29 (Full uniform positivity)**.** *There is a full Deninger cohomology theory H as in Conjecture 1.27 that is polarized.*

A full Deninger cohomology theory is in particular a Deninger cohomology theory, so Conjecture 1.27 implies Conjecture 1.26, and Conjecture 1.29 implies Conjecture 1.28. Each positivity conjecture implies the corresponding realization conjecture.

These conjectures concern the trivial coefficient and have $w_X = 0$. Nontrivial coefficients require additional spectral realizations, external products and duality, and arithmetic Abel–Jacobi comparisons.

Together Conjectures 1.26 and 1.28 assert the existence of a polarized Deninger cohomology theory with the finite-field comparison, and their full versions assert a polarized full one. The positive Hodge-star clause is the one in [FM20, Conjecture 2.3(iv), p. 522] used by Theorem 1.15. The positive form must be compatible with the same generator that realizes the zeta divisor. It plays the role of the positivity supplied by the Hodge standard conjecture, together with the Lefschetz formalism; it is not a claim that Grothendieck's standard conjectures already apply to an unconstructed arithmetic cohomology.

Theorem 1.30 (Riemann hypothesis for number fields)**.** *Let K be a number field. If Λ_K has a polarized graded spectral realization of type $(1, 0)$ with endpoint data $(V^0, \Theta_0) = (\mathbb{C}, 0)$ and $(V^2, \Theta_2) = (\mathbb{C}, 1)$, every nontrivial zero of ζ_K has real part $1/2$. If such realizations exist for all number fields, every zero in the critical strip of every finite-order Hecke L-function, and every zero or pole there of every Artin L-function, has real part $1/2$.*

Proof. The endpoint data and the divisor identity identify the spectrum in degree one with the divisor of $s(s-1)\Lambda_K(s)$, where Λ_K is the completed Dedekind zeta function (Definition 1.16). Apply Theorem 1.15 with $d = 1$, $w = 0$. The Hecke and Artin conclusions follow by the Artin formalism, with the full proof in Theorem 7.4. □

Conjectures 1.26 and 1.28 imply this theorem's hypotheses on the Arakelov objects. An absolute or characteristic-one model with the specified realization gives the identical implication.

For $K = \mathbb{Q}$ this is the Riemann hypothesis for ζ. No simplicity of zeros or holomorphy of nontrivial Artin L-functions is asserted. The existence of a positive spectral model for a single completed Dedekind zeta function is equivalent to its Riemann hypothesis (Proposition 7.7); an objectwise construction from the zeros alone does not supply the asserted functorial theory or an independent proof of its positivity.

The finite-field counterpart, with its full hypotheses and proof, is Theorem 7.24. Its logarithmically suspended cohomology satisfies the spectral and duality axioms; the positive form exists precisely when Frobenius is semisimple and pure. Purity is Deligne's theorem, whereas general semisimplicity is not asserted. For curves, the Rosati involution supplies the required positivity (Proposition 4.20).

1.4. **Base, source, and existing constructions.** This subsection compares Definition 1.21 with the Weil cohomology formalism and with the existing constructions, and explains why the source is $\overline{\mathrm{Sch}}$ rather than a category of schemes over a fixed base. The definitions of Flow, Deninger structures, polarizability, and Deninger modules are in Section 1.2, and the definition of a Deninger cohomology theory is in Section 1.3.

It is natural to expect F2 to be the analogue of the Lefschetz trace formula axiom of a Weil cohomology theory. In fact Weil cohomology theories have no such axiom: they are defined on smooth projective varieties over an algebraically closed field by finiteness, Poincaré duality, Künneth, and cycle-class axioms [Kle68, §1.2, pp. 362–363], and the trace formula is a consequence [Kle68, Proposition 1.3.6, p. 366]. Axiom F2 instead refers to a function given independently of H, because no point-counting primitive is available on the cohomological side; its dynamical form is AW6′ (Theorem 7.43). Together with Proposition 1.10((b)), axiom F3 propagates divisor identities along stratifications, as $\zeta(X, s) = \zeta(W, s)\zeta(Y, s)$.

It is natural to expect a single class of admissible objects on which every axiom is imposed. In fact [FM20, Conjecture 2.3, p. 522] imposes its parts on three classes: the grading and determinant formula on all arithmetic schemes and their compactifications with compact supports, the trace and perfect pairing for regular schemes of pure dimension, and the Hodge star, positivity, and the location of zeros for projective regular ones. The filtration $\overline{\mathrm{Sch}} \supset \overline{\mathrm{Sch}}^{\mathrm{reg}} \supset \overline{\mathrm{Sch}}^{\mathrm{proj}}$ of Definition 1.19 records these classes. Projective regular objects play the role of smooth projective varieties.

Proposition 1.31 (Schemes over $\mathcal{O}_K$ form a non-full subcategory)**.** *Let K be a number field. The forgetful functor from schemes over $\operatorname{Spec}\mathcal{O}_K$ to schemes is faithful and not full if $\operatorname{Aut}(K) \neq 1$. The automorphism group of $\operatorname{Spec}\mathcal{O}_K$ over $\operatorname{Spec}\mathcal{O}_K$ is trivial, while its automorphism group as a scheme is $\operatorname{Aut}(K)$.*

Proof. Faithfulness holds because a morphism over $\operatorname{Spec}\mathcal{O}_K$ is in particular a morphism of schemes. The scheme automorphisms of $\operatorname{Spec}\mathcal{O}_K$ are the $\operatorname{Spec}\sigma$ for ring automorphisms σ of $\mathcal{O}_K$, which are the restrictions of the elements of $\operatorname{Aut}(K)$. A morphism over $\operatorname{Spec}\mathcal{O}_K$ commutes with the identity structure map, so the only automorphism over $\operatorname{Spec}\mathcal{O}_K$ is the identity. □

It is natural to expect a separate theory for each number field K, defined on schemes over $\operatorname{Spec}\mathcal{O}_K$. In fact such a family loses the morphisms that are not $\mathcal{O}_K$-linear, by Proposition 1.31. These include the Galois actions used in Deninger's formulation over all number fields [Den25, Prediction 2.13, p. 8] and in the obstruction to real coefficients [Den23, 2.9, p. 4, and Theorem 2.13, p. 5]. Since $\operatorname{Spec}\mathbb{Z}$ is terminal among schemes, every scheme is a scheme over $\operatorname{Spec}\mathbb{Z}$, and Definition 1.21 fixes no base. A theory relative to $\mathcal{O}_K$ is the restriction of H with the algebra structure of Proposition 1.22.

Proposition 1.32 (Zeta functions do not depend on a base)**.** *Let X be a separated scheme of finite type over $\operatorname{Spec}\mathbb{Z}$, and let X_0 be its set of closed points. Then $\zeta(X,s)=\prod_{x\in X_0}(1-N(x)^{-s})^{-1}$ is the same whether X is regarded as a scheme over $\operatorname{Spec}\mathbb{Z}$ or over $\operatorname{Spec}\mathcal{O}_K$ for a structure morphism $X\to\operatorname{Spec}\mathcal{O}_K$.*

Proof. The closed points of X and the cardinalities $N(x)$ of their residue fields are determined by the scheme X. □

Proposition 1.33 (Weil restriction changes the zeta function)**.** *Let K be a number field of degree $n\geq 2$. Then $\operatorname{Res}_{\mathcal{O}_K/\mathbb{Z}}\mathbb{A}^1_{\mathcal{O}_K}\cong\mathbb{A}^n_{\mathbb{Z}}$, $\zeta(\mathbb{A}^1_{\mathcal{O}_K},s)=\zeta_K(s-1)$, and $\zeta(\mathbb{A}^n_{\mathbb{Z}},s)=\zeta(s-n)$, and these two functions are different.*

Proof. The Weil restriction represents $T\mapsto\mathcal{O}_K\otimes_{\mathbb{Z}}\Gamma(T,\mathcal{O}_T)$, and $\mathcal{O}_K$ is free of rank n over $\mathbb{Z}$, so it is $\mathbb{A}^n_{\mathbb{Z}}$. For a scheme Y of finite type over $\mathbb{Z}$, the fibre of $\mathbb{A}^1_Y$ over a closed point y is $\mathbb{A}^1_{\kappa(y)}$, with zeta function $(1-N(y)^{1-s})^{-1}$; hence $\zeta(\mathbb{A}^1_Y,s)=\zeta(Y,s-1)$, and likewise $\zeta(\mathbb{A}^n_{\mathbb{Z}},s)=\zeta(s-n)$. The first function has its rightmost pole at $s=2$, and the second at $s=n+1\geq 3$. □

It is natural to expect a theory relative to $\mathcal{O}_K$ to be carried to $\operatorname{Spec}\mathbb{Z}$ by Weil restriction. In fact, by Proposition 1.33, Weil restriction changes the prescribed function, while by Proposition 1.32 the forgetful functor preserves it. The zeta-preserving passage from schemes over $\mathcal{O}_K$ to $\overline{\mathrm{Sch}}$ is the forgetful functor.

Proposition 1.34 (The suspension does not depend on the base field)**.** *Let X be a smooth projective geometrically connected variety over $\mathbb{F}_{q^n}$, and let X' be the same scheme regarded as a variety over $\mathbb{F}_q$. Let $\Sigma_q(X')$ be the logarithmic suspension of $H^\bullet(X'_{\overline{\mathbb{F}}_q},\mathbb{Q}_\ell)\otimes\mathbb{C}$ formed with the logarithms base q of the eigenvalues of the geometric q-Frobenius, and $\Sigma_{q^n}(X)$ the suspension of $H^\bullet(X_{\overline{\mathbb{F}}_q},\mathbb{Q}_\ell)\otimes\mathbb{C}$ formed with the logarithms base q^n of the eigenvalues of the geometric q^n-Frobenius. Then $\Sigma_q(X')\cong\Sigma_{q^n}(X)$ as graded spectral cores, and $\Sigma_q(X')^0\cong\mathbb{K}_{q^n}$.*

Proof. Since $\mathbb{F}_{q^n}\otimes_{\mathbb{F}_q}\overline{\mathbb{F}}_q\cong\overline{\mathbb{F}}_q^{\,n}$, the Galois module $H^j(X'_{\overline{\mathbb{F}}_q})$ is induced from $H^j(X_{\overline{\mathbb{F}}_q})$ along $G_{\mathbb{F}_{q^n}}\subset G_{\mathbb{F}_q}$. If β is an eigenvalue of Fr_{q^n} with multiplicity m, the characteristic polynomial of Fr_q on the induced module contains the factor $(T^n-\beta)^m$, so the eigenvalues of Fr_q over β are the n roots $\beta^{1/n}\zeta_n^k$, each with multiplicity m. Their logarithms base q are the numbers $(\log\beta+2\pi i(k+nl))/(n\log q)$ for $0\leq k<n$ and $l\in\mathbb{Z}$, that is the numbers $(\log\beta+2\pi ij)/\log(q^n)$ for $j\in\mathbb{Z}$, which are the logarithms base q^n of β, with the same multiplicities. In degree zero $\beta=1$, and the eigenvalues $2\pi ij/\log(q^n)$ are those of $\mathbb{K}_{q^n}$. □

It is natural to expect the base of a relative theory, $\operatorname{Spec}\mathcal{O}_K$ or $\operatorname{Spec}\mathbb{Z}$, to play the role of the base field $\mathbb{F}_q$. In fact the base field enters the axioms only through the unit U_X of F1, the cohomology of the absolute point, and by Proposition 1.34 even that unit is determined by the variety and not by a chosen base field. The cohomology of $\overline{X}_K$ has $H^0=\mathbb{C}$, H^1 infinite-dimensional, and $H^2=\mathbb{C}(-1)$ [Den98, (3), p. 168]: this is the shape of the cohomology of a curve over $\mathbb{F}_q$, not of a unit. Schemes over $\mathcal{O}_K$ are the analogue of varieties over a curve, and [Den98, p. 172] describes the relative structure of Proposition 1.22 as a module structure over the cohomology of the base.

It is natural to expect the Flach–Morin and Connes–Consani–Marcolli constructions to stand to Definition 1.21 as ℓ-adic and crystalline cohomology stand to the Weil cohomology axioms, which both satisfy [Kle68, §1.2, pp. 362–363, and Examples 1.2.5, p. 364]. In fact neither is an instance. The Flach–Morin complexes are real complexes at a fixed integer twist without a flow; their relation to the flow is [FM20, Conjecture 4.1, p. 527, and Remark 4.15, p. 535], as recorded in Remark 1.25. Their pullbacks are constructed for equidimensional flat maps of

regular proper arithmetic schemes [FM18, §6.6, p. 1522], and the compactified complexes depend on the Kähler form [FM20, §3.2, p. 523]. The adelic cyclic homology of [CCM09, Definition 4.10, p. 21] is a degree-one space attached to a global field, without schemes or cup products, and its functoriality for Galois extensions passes through a non-canonical covering [CCM09, §9, pp. 50–54]. Over number fields no instance of Definition 1.21 is known. Over finite fields the logarithmic suspension of ℓ-adic cohomology satisfies the corresponding spectral and duality axioms (Theorem 7.24). What the existing constructions share is the degree-one spectral datum of the objects of $\overline{\mathrm{Sch}}_1$ together with the finite-field suspension.

1.5. **The absolute-base philosophy and the search outcomes.** The $\mathbb{F}_1$ philosophy asks what geometry would let arithmetic use the same square-and-correspondences argument as a curve over a finite field. For a smooth projective curve $C/\mathbb{F}_q$, the surface $C \times_{\mathbb{F}_q} C$ carries the diagonal and the graphs of powers of Frobenius. Their intersections encode point counts, and the positivity of correspondences, expressed by the Hodge index theorem or the Rosati involution, supplies Weil's bound (Section 4.3). The closed points of $\operatorname{Spec}\mathbb{Z}$ are the primes, with Euler factors $(1-p^{-s})^{-1}$. The analogy calls for a geometric setting in which the arithmetic curve has an equally useful square and correspondences.

An absolute base is meant to make that arithmetic square possible. The ordinary product

$$\operatorname{Spec}\mathbb{Z} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Z} = \operatorname{Spec}\mathbb{Z}$$

does not produce a surface. The notation $\operatorname{Spec}\mathbb{Z}\times_{\operatorname{Spec}\mathbb{F}_1}\operatorname{Spec}\mathbb{Z}$ expresses the desired replacement in a proposed geometry over a base below $\operatorname{Spec}\mathbb{Z}$; it is not a fibre product of ordinary schemes over a one-element field. Manin's absolute motives and tensor products of zeta functions make this motivation explicit [Man95, p. 121, p. 132, and §5.2, p. 158]. Adding the infinite places to obtain $\overline{\operatorname{Spec}\mathbb{Z}}$ addresses a second requirement, the completion of the arithmetic curve and its zeta function. It does not itself specify this absolute base or its products. A uniform treatment must say both what the compactified objects are and how their proposed geometry supplies products.

Remark 1.35 (There is no single $\mathbb{F}_1$ geometry). It is natural to speak of "the" category of schemes over $\mathbb{F}_1$. In fact several such categories have been constructed for different purposes, and no single category is known to which all of them are equivalent. Among them are Deitmar's monoid schemes [Dei04], Soulé's varieties [Sou03], the relative schemes of Toën–Vaquié [TV05], Durov's generalized rings [Dur07], Borger's Λ-schemes [Bor09], Lorscheid's blue schemes [Lor11], the glued schemes of Connes and Consani [CC09], and their arithmetic and scaling sites [CC14a, CC15]; surveys are [PL09] and [Lor18, p. 1]. Some comparisons are theorems: Deitmar's monoid schemes are equivalent to the Toën–Vaquié schemes relative to sets [Vez10, Theorem 36]. Others are not equivalences. The arithmetic square also behaves differently: in Durov's theory $\mathbb{Z}\otimes_{\mathbb{F}_1}\mathbb{Z} = \mathbb{Z}$ [Dur07, 0.5.19, p. 36], and in Λ-geometry the product $\operatorname{Spec}\mathbb{Z}\times_{\operatorname{Spec}\mathbb{F}_1}\operatorname{Spec}\mathbb{Z}$ is $\operatorname{Spec}\mathbb{Z}$ (Proposition 1.36), while the arithmetic site has a square of a different kind [CC16, §6, p. 22]. A cohomological formulation therefore cannot take its source to be "the" $\mathbb{F}_1$ category; an absolute geometry enters through a chosen comparison functor from $\overline{\mathrm{Sch}}$, as in Section 1.3.

Proposition 1.36 (Products over $\mathbb{F}_1$ in Λ-geometry)**.** *In the Λ-algebraic geometry of* [Bor09]*, let $\mathbb{F}_1$ denote $\mathbb{Z}$ with its unique Λ-structure. Then* $\operatorname{Spec}\mathbb{Z}$ *is a terminal Λ-scheme, and* $\operatorname{Spec}\mathbb{Z}\times_{\operatorname{Spec}\mathbb{F}_1}\operatorname{Spec}\mathbb{Z} = \operatorname{Spec}\mathbb{Z}$ *in the category of Λ-schemes.*

Proof. By [Bor09, p. 2], $\mathbb{Z}$ with its Λ-structure is the initial object in the category of Λ-rings, so its spectrum is terminal among Λ-schemes. A product with a terminal object is the other factor. □

It is natural to expect the source of a Deninger cohomology theory to be a category over $\operatorname{Spec}\mathbb{F}_1$ containing the arithmetic square. In fact such a square cannot be an object on which the theory takes values in spectral cores (Theorem 8.12), in Λ-geometry the square collapses (Proposition 1.36), and by Remark 1.35 there is no single $\mathbb{F}_1$ category to choose. The source of Definition 1.21 is therefore $\overline{\mathrm{Sch}}^{\mathrm{proj}}$, and a geometry containing the square is an ambient category reached by a comparison functor.

The search found precise failures of several expected properties of such a square. An untwisted tensor formula over $\operatorname{Spec}\mathbb{Z}$ is incompatible with the spectral normalization of the arithmetic curve: it would identify its degree-one flow module with two copies of itself (Proposition 8.5). General derived tensor formulas over arbitrary bases also fail in the finite-field examples of Propositions 8.6 and 8.9. These results delimit particular Künneth formulas; they do not rule out an absolute product with a different base and comparison theorem.

Even an absolute square cannot inherit every analytic property of the curve if it retains the expected cross products. Under the spectral realization, duality, pullback, cup-product, and multiplicative-trace hypotheses of Theorem 8.12, the cross-product map on degree-one tensors is injective. For a square, the pairs of zeros ρ and $1-\rho$ then give infinitely many independent vectors with flow eigenvalue 1. Thus its degree-two flow module fails the finite multiplicity requirement for the spectral cores used here. Under the positive Hilbert-space realization, the corresponding tensor-sum operator has no compact resolvent (Proposition 8.16). The infinite multiplicity needs no extra hypothesis on zero spacings; the stronger accumulation claim in Theorem 8.12 has its own stated hypothesis. Deninger and Manin had already identified the sum-of-zeros difficulty [Den94, (7.32), pp. 41–42], [Man95, p. 132].

Restricting tensor pairs to imaginary parts of the same sign does not supply the missing cohomology ring. This restriction, Kurokawa's cone, excludes the tensors pairing nonreal ρ with $1-\rho$. An algebra containing the pullbacks from both factors must contain all their cross products, including these excluded tensors (Proposition 8.13). The periodic-orbit model exhibits another difficulty: the tensor sum for distinct prime periods $\log p$ and $\log p'$ has dense imaginary spectrum, whereas a repeated prime period gives infinite multiplicities (Proposition 8.18). The corresponding diagonal flows have, respectively, no closed orbits or nonisolated families of them (Proposition 8.19); a product trace formula cannot simply retain the isolated-orbit description of the factors.

Connes–Consani supply concrete characteristic-one geometry in which to pursue the absolute-base idea. Characteristic one here refers to idempotent addition, as in Boolean and tropical semirings, rather than to an ordinary field. Their arithmetic site is a semiringed topos, and their scaling site extends its scalars from the Boolean semifield to the tropical semifield. The arithmetic site has a constructed square and Frobenius correspondences [CC16, §6]. In the version of the site used in [CC14a, Theorems 2.6–2.7], points over the maximal compact subsemiring $[0,1]_{\max}$ of the tropical semifield recover an adelic quotient, and the counting distribution gives the completed Riemann zeta function [CC14a, Theorems 2.6–2.7]. Thus the search has actual geometric objects and zeta identities to compare, not only a proposed notation for an absolute square.

The available Riemann–Roch theorems identify a further part of the analogy that has been realized. On the scaling site's periodic orbit at a prime p, whose length is $\log p$, Connes–Consani construct divisors and prove a Riemann–Roch formula (Theorem 11.23). Their later Riemann–Roch theorem on $\overline{\operatorname{Spec}\mathbb{Z}}$ is also a construction in dimension one [CC23, Theorem 1.1, p. 2]. Neither statement is the global intersection inequality on a square needed for the RH strategy. The checked sources propose that further step through a Riemann–Roch and intersection-theoretic framework [CC19, §3, pp. 16–18, and p. 22].

The search did not find a comparison identifying these constructions with a single cohomology of the compactified arithmetic curve that has the required global zeta spectrum, duality, and positive star. Connes–Consani's archimedean cyclic cohomology realizes gamma factors (Theorem 11.8); it does not by that theorem realize the nontrivial zeros. Ayoub's existing realization theories and the known Weil–Arakelov constructions likewise supply specific comparison results, whose scope is listed in Table 8. The missing global identification and positivity theorem are open requirements in this synthesis. Their absence from the checked corpus is a search outcome, not a nonexistence theorem for $\mathbb{F}_1$ cohomology.

The finite-field comparison also produces a constructive correction. After replacing Frobenius by its full logarithmic suspension, even the finite-field product formula requires a relative tensor product over the periodic unit $\mathbb{1}_q$, the suspended cohomology of the base. Tensoring instead over $\mathbb{C}$ repeats every spectral value infinitely (Proposition 8.17); tensoring over $\mathbb{1}_q$ gives the actual Künneth isomorphism of Theorem 8.23. The choice of base and tensor product is therefore part of the geometric proposal, including in the case where the cohomology already exists.

The surviving proposal separates the curve's zeta realization from the larger space needed for its square. Conjecture 1.40 requires a multiplicative surface datum and a Künneth isomorphism, with a Hodge-index condition on the tensors pairing eigenvalues ρ and $1-\rho$. It imposes no finite-multiplicity, determinant, or compact-resolvent condition on that surface. With its stated duality and star compatibilities, the Hodge-index clause gives purity and hence RH by Theorem 8.34. This is the role analogous to a standard conjecture: the required positivity is a geometric statement about correspondences on a square. A construction realizing these data in the arithmetic or scaling site, together with the comparison identifying its intersection form, remains to be supplied.

In the uniform formulation below, compactified arithmetic schemes, finite-field varieties, and a specified absolute geometry can provide inputs to the same spectral implication. The objects of $\overline{\mathrm{Sch}}^{\mathrm{proj}}$, on which a zeta determinant is required, do not include absolute squares. This distinction is forced by the tensor obstruction above and permits a larger target for the surface cohomology. The $\mathbb{F}_1$ part of the search therefore contributes both a geometric construction problem and restrictions on its possible solution: realize the required cohomology on an appropriate absolute square, compare its pairings with Weil positivity, and prove the compatible positive structure. The manuscript reports those restrictions together with the constructions that are already available.

1.6. **Comparison with the Weil and standard conjectures.** Over a finite field, ℓ-adic cohomology is already constructed; the Lefschetz and Hodge standard conjectures seek algebraic correspondences and positivity on it. Their conjunction gives the Riemann hypothesis by Serre's argument [Kle94, Theorem 5-6, p. 19]. In the arithmetic setting both the global spectral cohomology and its positive structure are conjectural. Ayoub's Weil theory, Connes–Consani's cyclic homology and arithmetic sites, and Weil–Arakelov complexes are actual constructions with different coefficient rings, source categories, and comparison theorems. Table 8 states those differences and the remaining conjectures separately.

The analogy between Theorem A and Theorem B is exact for positivity and duality, and not symmetric in one place. Over $\mathbb{F}_q$ the cohomology is constructed independently of the zeta function, and rationality and AW1–AW3 are theorems about it, rationality being derived from the Lefschetz trace formula for all powers of Frobenius (Theorem 7.45(iii); in these notes rationality itself is imported, Theorem 3.8, and the Lefschetz formula is read off from it); over a number field no object is known that satisfies AW1 with an arithmetically defined cup product, and the divisor identity AW2 is imposed as an axiom. Given AW1, AW2 is equivalent to the trace-formula axiom AW6′, whose geometric side is the explicit formula of ζ_K (Definition 5.9,

Theorem 7.43): imposing AW6′ instead of AW2 changes the form of the assumption, not its strength. Test functions supported in a bounded interval $(0, L)$ of positive times do not determine the spectrum (Theorem 7.50); every tail of times does. The determinant normalization, exactly one over $\mathbb{F}_q$ (Theorem 7.20), is computed over every number field: it is $2^{-n/2}|D_K|^{-s/2}$ relative to Λ_K and exactly one relative to Deninger's completion (Theorem 7.37). Table 1 is the dictionary between the two settings.

Table 1: Dictionary between Theorem B over $\mathbb{F}_q$ and Theorem A over number fields.

Smooth projective $X/\mathbb{F}_q$	Number field K, object $\overline{X}_K$	Where
existence of $H^\bullet$: a theorem (Grothendieck), independent of $Z(X,t)$	existence of a Weil datum from arithmetic input: open	Thm. 4.9, Sec. 13
geometric Frobenius F on H^j	flow $e^{t\Theta}$ with generator Θ_j on a spectral core V^j; $\mathrm{F} = q^{\Theta}$ on the suspension	Def. 1.2, 4.15
eigenvalue α of F	the progression $\Sigma_q(\alpha)$ of eigenvalues ρ of Θ	Def. 4.14
$\lvert\alpha\rvert = q^{j/2}$	$\operatorname{Re}\rho = j/2$ (for K: $\operatorname{Re}\rho = \frac{1}{2}$)	Prop. 3.3
rationality: $Z(X,t) = \prod_j \det(1 - \mathrm{F}t \mid H^j)^{(-1)^{j+1}}$, a theorem derived from the Lefschetz trace formula	AW2: $\operatorname{div}\Lambda_K = \sum_j (-1)^{j+1}\operatorname{Sp}_j$, imposed as an axiom, equivalent given AW1 to AW6′; AW-det, with normalization term $2^{-n/2}\lvert D_K\rvert^{-s/2}$, and 1 for $\widehat{\zeta}_K$	Def. 1.4, 5.6, Thm. 7.43, 7.37
Poincaré duality, $\mathrm{F} = q^d$ on H^{2d}	AW3: duality pairing ρ with $d - \rho$	Rem. 5.10
unit H^0 with $\mathrm{F} = 1$; periodic unit $\mathbb{1}_q$ after suspension	arithmetic unit $\mathbb{1} = (\mathbb{C}, 0)$; $D_0 = s/2\pi$	Prop. 7.23, Sec. 7.7
positive Frobenius polarization $h(\mathrm{F}u, \mathrm{F}v) = q^j h(u,v)$	positive infinitesimal polarization $h(\Theta u, v) + h(u, \Theta v) = j h(u,v)$ (AW4-V)	Def. 4.1, 1.7
Weil/Rosati positivity: $\operatorname{Tr}(\alpha\alpha^\dagger) > 0$ on $\operatorname{End}^0(J)$ with $\pi\pi^\dagger = q$, the multiplicative algebra form AW4-E_F	positive trace involution on an algebra containing Θ with $\Theta^\dagger = j - \Theta$ (AW4-E), which AW4-E_F implies on the suspension	Prop. 4.8, 4.19, 4.20, 6.2
Castelnuovo positivity of correspondences on $C \times C$ (Hodge index)	Hodge index clause on the algebraic classes of $\overline{X}_K \boxtimes \overline{X}_K$: equivalent to purity with definite star forms	Thm. 8.34, Conj. 1.40
Künneth: $H^\bullet(X \times Y) = H^\bullet(X) \otimes H^\bullet(Y)$; after suspension, a tensor product over $\mathbb{1}_q$	conjectural; the tensor square is not a spectral core, and there is no formula over $\operatorname{Spec}\mathbb{Z}$	Thm. 8.23, 8.12, Prop. 8.5
suspended star, $\operatorname{tr}(u \cup *_j v) = h_j(u,v)$, $*_j\mathrm{F} = q^{j-d}\mathrm{F}*_j$ (when F is semisimple and pure)	AW4: positive flow-compatible star, $\Theta_{2d-j}*_j = *_j\Theta_j + (d-j)*_j$	Def. 7.18, Prop. 6.1

Smooth projective $X/\mathbb{F}_q$	Number field K, object $\overline{X}_K$	Where
positive form exists ⇔ F semisimple and pure	package on one object ⇔ RH for that object	Prop. 4.3, 7.7
Lefschetz trace formula for all F^n, a theorem, from which rationality follows; two-sided form	the trace-formula axiom AW6′ (explicit formula of ζ_K), an axiom, equivalent to AW2 given AW1; one-sided bounded windows of times do not suffice	Def. 5.9, Thm. 7.45, 7.43, 7.50
RH: Weil (curves, via positivity), Deligne (all X, not via positivity); the positivity axiom known where F is semisimple	the axioms: open	Rem. 7.25, Sec. 13

1.7. **Other obstructions found by the search.** *What fails.* Sections 6.1 and 9 record seventeen findings (R1–R17, correcting the expectations O01–O23 of the register in Section 9.5) about expectations that are natural when intuition is imported from the finite-field case and that are false. The Weil formalism, perfect duality, and a flow-compatible quaternionic star do not force purity (Propositions 6.9–6.12); comparisons at integer twists do not see a nonreal spectrum (Proposition 9.3); skew-symmetry does not control a spectrum (Proposition 9.6); real coefficients, general leafwise cohomology, and local determinant formulas each fail for a precise reason (Remarks 9.21 and 9.24, Propositions 9.22 and 9.26); scalar de Branges models do not see zero multiplicities, and several positivity statements that look sufficient are not (Propositions 9.8–9.20 and Remark 9.30); and the literal formulas of one version of a recent comparison produce a degenerate seminorm (Proposition 9.34).

Sixteen further corrected expectations (O24–O39), listed in the register of Section 9.5, concern the main theorem and its calibration: the axioms other than positivity can be satisfied with any spectrum symmetric under $\rho \mapsto 1-\rho$ with even central multiplicity, and a flow-compatible form that is not positive allows zeros off the line (Proposition 7.7); Galois descent controls only the rational orbits of isotypic divisors (Proposition 7.12); the zeta function of $\mathbb{P}^1_{\mathbb{Z}}$ has zeros on two lines (Proposition 7.27), and without positivity the total zeta function of an arithmetic surface does not determine its degreewise spectra (Proposition 7.30); over $\mathbb{F}_q$, a "formal curve" with Poincaré duality, rational zeta function, functional equation, and integral point counts violates purity (Proposition 4.6), Weil's positivity is not a form on H^1 (Proposition 4.12), a single branch of the logarithm destroys the divisor (Proposition 7.22), and the unit object of the axioms must be the periodic cohomology of the base (Proposition 7.23).

Seventeen more (O40–O56) concern the multiplicative form of Weil's positivity (Proposition 4.19), the parity of central orders (Proposition 7.9), and the Künneth conjecture (Section 8).

Seventeen further (O57–O73) concern the normalization of the determinant over a number field and the trace-formula axiom (Sections 7.7 and 7.8), and two more (O74–O75) the Eilenberg–Moore comparison over a base (Section 8).

1.8. **The construction and positivity problems.** For a single number field, AW1–AW3 can be satisfied by assigning vector spaces to the known zeta zeros. The full objectwise package exists if and only if RH holds for that Dedekind zeta function (Proposition 7.7). Such a model does not construct cohomology from arithmetic geometry or establish comparisons with existing theories. The research problem is to construct the spaces, products, and pairings independently of the zeros and then prove that the same realization has the compatible positivity of AW4.

The distinction persists in the other cases considered here. For one Galois extension, the package with descent follows from the relevant Riemann hypotheses together with Artin holomorphy (Proposition 7.14). Over a finite field the positive Frobenius-compatible form requires both purity and semisimplicity. For higher-dimensional arithmetic objects, the resulting zero and pole lines depend on cohomological degree (Theorem 7.28).

1.9. **The Künneth and Hodge-index conjecture.** Weil's argument uses correspondences on $C \times C$. To state its arithmetic analogue, the following definitions allow a square with larger cohomology spaces than those of the curve. The symbol $\overline{X}_K \boxtimes \overline{X}_L$ denotes the intended arithmetic square; it does not specify an ordinary fibre product over $\operatorname{Spec}\mathbb{Z}$.

Definition 1.37. A Weil datum $(V^\bullet, \Theta, \cup, \mathrm{Tr}, *)$ of type $(1,0)$ satisfies the *algebraic part of AW4 in degree one* if $*_1$ is bijective, $\Theta_1 *_1 = *_1 \Theta_1$, $B_1(*_1 x, *_1 y) = \overline{B_1(x,y)}$ for all $x, y \in V^1$, and the star form h_1 is Hermitian.

Definition 1.38. Let $D_K = (V_K^\bullet, \Theta, \cup, \mathrm{Tr}_K, *)$ and $D_L = (V_L^\bullet, \Theta, \cup, \mathrm{Tr}_L, *)$ be Weil data of type $(1,0)$ on $\overline{X}_K$ and $\overline{X}_L$.

(a) A *surface datum* over (D_K, D_L) consists of an associative graded-commutative unital graded algebra $H^\bullet = \bigoplus_{j=0}^4 H^j$ over $\mathbb{C}$, a degree-preserving derivation Θ_S of it, a linear *trace* $\mathrm{Tr}_S : H^4 \to \mathbb{C}$, antilinear *stars* $*_S : H^j \to H^{4-j}$, and graded unital algebra homomorphisms $\mathrm{pr}_1^* : V_K^\bullet \to H^\bullet$ and $\mathrm{pr}_2^* : V_L^\bullet \to H^\bullet$ with $\Theta_S \mathrm{pr}_i^* = \mathrm{pr}_i^* \Theta$.
(b) Its *Künneth map* is $\kappa : V_K^\bullet \otimes V_L^\bullet \to H^\bullet$, $\kappa(x \otimes y) = \mathrm{pr}_1^* x \cup \mathrm{pr}_2^* y$.
(c) Its *algebraic part* is $N_S = \kappa\big(\bigoplus_\rho V^1_{K,\Theta\sim\rho} \otimes V^1_{L,\Theta\sim 1-\rho}\big) \subset H^2$.
(d) Its *star forms* are $h_S^j(u,v) = \mathrm{Tr}_S(u \cup *_S v)$ on H^j.
(e) It is *multiplicative* if $\mathrm{Tr}_S(\mathrm{pr}_1^* x \cup \mathrm{pr}_2^* y) = \mathrm{Tr}_K(x)\,\mathrm{Tr}_L(y)$ for $x \in V_K^2$, $y \in V_L^2$, and $*_S \kappa(x \otimes y) = (-1)^{|x||y|}\kappa(*x \otimes *y)$ for homogeneous x, y.
(f) It satisfies the *Hodge index clause* if h_S^2 is positive definite on N_S.

The Hodge-index clause is positivity of the star form on N_S. It is not positivity of the intersection pairing on all of H^2. A geometric construction must compare these tensors and their form with actual correspondences on the proposed arithmetic square.

Definition 1.39. Let $D = (V^\bullet, \Theta, \cup, \mathrm{Tr}, *)$ and $\widetilde{D} = (\widetilde{V}^\bullet, \widetilde{\Theta}, \cup, \widetilde{\mathrm{Tr}}, \tilde{*})$ be Weil data of types $(d,0)$ and $(\tilde{d},0)$.

(a) The *tensor datum* $D \otimes \widetilde{D}$ has $(V \otimes \widetilde{V})^n = \bigoplus_{a+b=n} V^a \otimes \widetilde{V}^b$ with the flow $\Theta \otimes 1 + 1 \otimes \widetilde{\Theta}$, the product $(x \otimes y) \cup (x' \otimes y') = (-1)^{|y||x'|}(x \cup x') \otimes (y \cup y')$, the trace $\mathrm{Tr} \otimes \widetilde{\mathrm{Tr}}$ on $V^{2d} \otimes \widetilde{V}^{2\tilde{d}}$, and the stars $*(x \otimes y) = (-1)^{|x||y|} * x \otimes \tilde{*} y$ for homogeneous x, y.
(b) For $n \geq 0$, the *Tate datum* T_n is the Weil datum of type $(n,0)$ with $T_n^{2i} = \mathbb{C}h^i$ for $0 \leq i \leq n$ and $T_n^j = 0$ for odd j, generator $\Theta h^i = i h^i$, product $h^i h^k = h^{i+k}$ (zero if $i + k > n$), trace $\mathrm{Tr}(ch^n) = c$, and stars $*(ch^i) = \bar{c} h^{n-i}$.

Conjecture 1.40 (Künneth conjecture)**.** *There are, for every number field K, a Weil datum D_K of type $(1,0)$ on $\overline{X}_K$ satisfying AW1, AW2, and AW3 for $(\Lambda_K, \nVdash)$ and the algebraic part of AW4 in degree one (Definition 1.37), and, for every pair of number fields (K,L), a multiplicative surface datum $S_{K,L}$ over (D_K, D_L) (Definition 1.38) such that:*

(K1) the Künneth map of $S_{K,L}$ is an isomorphism of graded algebras, for the Koszul product on $V_K^\bullet \otimes V_L^\bullet$ (Definition 1.39);
(K2) $S_{K,K}$ satisfies the Hodge index clause.

No spectral-core, determinant, or compact-resolvent condition is imposed on $S_{K,L}$.

The curve data in this conjecture satisfy only the algebraic star condition before the surface positivity is imposed. Under the stated compatibilities, the Hodge-index clause gives purity and definiteness of the degree-one star form on each generalized eigenspace, and hence RH (Theorem 8.34). The signs of that form may depend on the eigenspace. No comparison identifying these data with the cohomology of the arithmetic or scaling site is asserted here.

The stronger product expectations have explicit obstructions. Ordinary products over $\operatorname{Spec}\mathbb{Z}$ do not satisfy the proposed untwisted Künneth formula (Proposition 8.5); the general derived comparison also fails without its additional hypotheses (Propositions 8.6 and 8.9). On an arithmetic self-square, duality forces infinite multiplicity, and a positive Hilbert realization has no compact resolvent (Theorem 8.12, Proposition 8.16). The same-sign cone omits required cross products (Proposition 8.13). Positive product results remain over the finite-field periodic unit and for Tate data (Theorem 8.23, Proposition 8.33).

Other results. Section 12 reproduces Lagarias's RH-equivalent Hermite–Biehler criterion, keeping multiplicities explicit (Proposition 12.1), and records what canonical systems and the Weil form do and do not supply.

What is imported. Section 3.1 states, with exact source locators and hypotheses, the classical analytic facts used; Sections 4 and 7 state the finite-field, Artin, and representation-theoretic facts where they are used; and Section 11 states in the same way the comparison theorems relevant to candidate theories (Section 10) and the de Branges and Weil-positivity results used in Section 12. None of them constructs the object of Theorem 7.4. The provenance of every other statement is recorded in Subsection 14.11.

What is open. Section 13 lists the open questions, including whether a trace formula tested on a two-sided bounded window of times determines the spectrum (Question 13.3); Subsection 14.13 records the Lean formalization, claim by claim.

2. Prior work

These notes are an exposition of previous work. Their central conditional statement, that a cohomology of the Arakelov compactification of $\operatorname{Spec}\mathcal{O}_K$ with a flow and a positive star compatible with it would force the zeros of ζ_K onto the critical line, is Deninger's formal argument, which he models on Serre's Kähler analogue of the Weil conjectures; its finite-field model is Weil's proof of the Riemann hypothesis for curves in the form given by Serre and Kleiman. This section surveys that work and its neighbours, with exact locators, and then records, statement by statement, where the results of the later sections appear in the literature or where their ideas come from. What the notes add is organization: one list of axioms with exact equivalences among their forms, a calibration of the axioms against the finite-field case, negative results and explicit counterexamples that delimit what the axioms other than positivity can give, and a Lean formalization of the linear-algebraic parts.

2.1. **Positivity and the Riemann hypothesis over finite fields.** Weil's proofs of the Riemann hypothesis for a curve C over $\mathbb{F}_q$ rest on one positivity statement: for a nonzero class ξ of correspondences on $C \times C$, the trace $\sigma(\xi \circ \xi')$ is positive, and applied to ξ built from the Frobenius graph it gives $|\alpha| = q^{1/2}$ for the Frobenius roots. It is stated with the deduction of the Riemann hypothesis in [Wei41, p. 346 and p. 347]; for the earlier note [Wei40] and for the book [Wei48a] see the account in [Mil15, §1]; Milne locates the positivity statement of the book as its Theorem 10, p. 54 [Mil15, p. 12]. In the language of abelian varieties the same positivity is that of the Rosati involution, $\operatorname{Tr}(\alpha\alpha^\dagger) > 0$ on $\operatorname{End}^0(J)$ ([Wei48b], as reported in [Mil15, p. 23]), [Mum74, §21, Theorem 1, p. 192], [Mil08, Theorem 14.3, p. 62], and the Riemann hypothesis follows by applying it to Frobenius, for which $\pi\pi^\dagger = q$ [Mil15, Theorems 1.27 and 1.30, pp. 21–22].

Serre proved the Kähler analogue: on a smooth complex projective variety, an endomorphism that multiplies a hyperplane class by q has eigenvalues of absolute value $q^{r/2}$ on H^r, because after normalization it is unitary for the positive definite Hermitian form given by the Hodge–Riemann bilinear relations; his second theorem is the higher-dimensional analogue of Castelnuovo's positivity [Ser60, Théorèmes 1 and 2, pp. 392–394]. This "unitary operator" pattern is the one transported by Grothendieck's standard conjectures and by Deninger's program. Kleiman and Grothendieck formulated the standard conjectures of Lefschetz type and of Hodge type and showed that together they imply the Riemann hypothesis with semisimple Frobenius by Serre's argument [Kle68, pp. 359–361, 3.11 pp. 380–381, Lemma 4.6 p. 385, Theorem 4.7 p. 386], [Gro69, Remark (2), p. 197], [Kle94, Theorem 5-6, p. 19]; both conjectures remain open in general. Deligne proved the Riemann hypothesis for smooth projective varieties over $\mathbb{F}_q$ by another route, through Lefschetz pencils, monodromy, and Rankin's method applied to tensor powers [Del74, Théorème (1.6), p. 276, and §3, pp. 283–284], and extended it to weights of mixed sheaves [Del80, Théorème 1, p. 138]; no positivity on algebraic cycles enters. The survey [Mil15] covers this history. Serre also observed that a Weil cohomology cannot have coefficients in $\mathbb{R}$, $\mathbb{Q}$, or $\mathbb{Q}_p$, because for a supersingular elliptic curve $\mathrm{End}(E)\otimes\mathbb{R}$ is Hamilton's quaternions [Mil15, (2.2) and fn. 25, p. 27]; this is the finite-field form of the obstruction of Section 2.4 below.

2.2. **Standard conjectures and Birch–Swinnerton-Dyer.** The comparison above suggests asking whether Grothendieck's standard conjectures also imply Birch–Swinnerton-Dyer. The phrase "standard conjectures" needs care here. Grothendieck's Lefschetz and Hodge standard conjectures concern algebraic correspondences and positivity on algebraic cycles of smooth projective varieties; they do not state a comparison between rational points and the central zero of an elliptic L-function [Gro69, pp. 193–197]. The divisor Tate conjecture on a surface over a finite field, and the conjectural arithmetic spectral and Abel–Jacobi maps for an elliptic motive, are different inputs. We record the supported implications and the exact gaps. The examples below refute particular shortcuts; they do not prove logical independence of BSD from all geometric standard conjectures.

Let E/K be an elliptic curve over a global field, and let $r=\operatorname{rank}E(K)$. Write

$$\mathrm{Sha}(E/K)=\ker\left(H^1(K,E)\longrightarrow\prod_v H^1(K_v,E)\right).$$

When $L(E/K,s)$ continues to $s=1$, put $r_{\mathrm{an}}=\operatorname{ord}_{s=1}L(E/K,s)$; here L includes the finite Euler factors at bad places. Rank BSD says $r_{\mathrm{an}}=r$. Full BSD also asserts finiteness of $\mathrm{Sha}(E/K)$ and identifies the first nonzero Taylor coefficient with period, regulator, local-component, and torsion factors. Over $\mathbb{Q}$, Wiles gives the refined formula in terms of these factors [Wil00, p. 2, Remark 1]. Its complete finite-place L-function is distinct from adjoining an archimedean factor in the spectral argument below. The general number-field formula has its own local measures.

Fix a prime $\ell\neq\operatorname{char}K$. The geometric divisor-class map used below is the base-changed first Chern class

$$c_1:\mathrm{Pic}(E)\otimes\mathbb{Q}_\ell\longrightarrow H^2_{\text{ét}}(E_{\overline{K}},\mathbb{Q}_\ell(1)).$$

Proposition 2.1. *Let E/K be an elliptic curve with origin O. Under the degree and O-splitting, $\mathrm{Pic}(E)\cong\mathbb{Z}\oplus E(K)$, while the geometric class map c_1 records only the degree factor. Consequently its kernel contains $E(K)\otimes_{\mathbb{Z}}\mathbb{Q}_\ell$.*

Proof. The degree map is split by $n\mapsto\mathcal{O}_E(nO)$, and $\mathrm{Pic}^0(E)(K)=E(K)$ by the Abel–Jacobi isomorphism for an elliptic curve. On $E_{\overline{K}}$, a divisor class in H^2 is its degree times the class of a point, so every degree-zero line bundle has zero cohomological class. Tensoring with $\mathbb{Q}_\ell$ gives the stated kernel. □

Remark 2.2. The examples in this subsection refute specific formal deductions from positivity, divisor data, or prime-primary information. They are not elliptic curves violating BSD, and they do not establish logical nonimplication from Grothendieck's standard conjectures imposed on all varieties. No such all-varieties implication is proved here.

The rational points sit in the homologically trivial part, not in the degree-one divisor class measured by the curve's standard-conjecture positivity. Over a function field, they can instead be represented by horizontal divisor classes on a fibred surface; that surface is the object in Theorem 2.6 below.

For the number-field spectral route, let $V_{E,(1)}$ denote a finite generalized 1-eigenspace of a proposed operator Θ for the elliptic motive. The relevant spectral point is 1, whereas the weight-zero zeta object of $\operatorname{Spec}\mathbb{Z}$ has central point $1/2$. Deninger's proposed arithmetic Abel–Jacobi map compares $E(K)\otimes\mathbb{C}$ with the central eigenspace [Den10, p. 8 and Example 5, p. 10]. For the conditional number-field statement below, write $\Lambda_E(s) = g(s)L(E/K,s)$ for a proposed completed factor; the proposition specifies the required behavior of g at 1.

Proposition 2.3. *For every $r \geq 0$ and $k > 0$ there are a positive Hermitian space V, an operator Θ, and an isometric injection $\mathbb{C}^r \hookrightarrow \ker(\Theta - 1)$ satisfying $\Theta + \Theta^* = 2$ but with $\dim\ker(\Theta-1) = r + k$.*

Proof. Take $V = \mathbb{C}^{r+k}$ with its usual Hermitian metric, $\Theta = 1$, and include the first r coordinates. The cokernel has dimension k. □

This is a linear-algebra counterexample to a positivity-only rank deduction, not an elliptic curve violating BSD. A positive height form does not make the arithmetic Abel–Jacobi map surjective.

Consider the Jordan matrix $J = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

Proposition 2.4. *For the matrix J just displayed, $\operatorname{ord}_{s=1}\det(s - J) = 2$, whereas $\dim\ker(J - 1) = 1$.*

Proof. The determinant is $(s-1)^2$, and $J - 1$ has rank one. □

Thus determinant multiplicity cannot be replaced by eigenspace dimension without semisimplicity or an equivalent block comparison.

Proposition 2.5 (Conditional rank BSD)**.** *Let E/K be an elliptic curve over a number field. Assume that: $L(E/K,s)$ is meromorphic near 1; the factor g in $\Lambda_E(s)$ is holomorphic and nonzero at 1; a spectral realization has a finite-dimensional generalized 1-eigenspace $V_{E,(1)}$ and the exact order identity $\operatorname{ord}_{s=1}\Lambda_E(s) = \dim V_{E,(1)}$; on this block a positive Hermitian form satisfies $\Theta + \Theta^* = 2$; and an arithmetic Abel–Jacobi map is an isomorphism*

$$E(K)\otimes_{\mathbb{Z}}\mathbb{C} \xrightarrow{\sim} \ker(\Theta - 1 \mid V_{E,(1)}).$$

Then $\operatorname{ord}_{s=1} L(E/K,s) = \operatorname{rank} E(K)$.

Proof. The factor g is a unit at 1, so it does not change the order. On the generalized block, $A = \Theta - 1$ is nilpotent. The positive form and $\Theta + \Theta^* = 2$ make A skew-Hermitian, hence diagonalizable; thus $A = 0$. The assumed order identity now gives

$$\operatorname{ord}_{s=1} L(E/K,s) = \dim V_{E,(1)} = \dim\ker(\Theta - 1) = \dim(E(K)\otimes\mathbb{C}) = r.$$

□

The exact order identity includes any noncancellation requirement for an alternating spectral determinant. This is the finite-dimensional form of Deninger's conditional argument [Den10, p. 8 and Example 5, p. 10]; it does not construct the spectral realization or the Abel–Jacobi

isomorphism. His printed elliptic example concerns $E/\mathbb{Q}$; the statement above for general K is the same conditional linear-algebra argument.

Deninger's coefficient cohomology and its determinant formula are proposals, and he presents the arithmetic Abel–Jacobi isomorphism as a further condition for rank BSD [Den10, pp. 6–8 and Example 5, p. 10]. The classical isomorphism $E(K) \cong \mathrm{Pic}^0(E)(K)$ used in Proposition 2.1 is a different map. The Néron–Tate height theorem gives a nondegenerate height pairing on $E(K) \otimes \mathbb{Q}$, but it neither constructs the proposed spectral target nor proves that an arithmetic map onto it is surjective. Once the spectral order identity is assumed, the dimension assertion in this last comparison is precisely the missing rank equality. Conjectures 1.26 and 1.28 concern trivial coefficients and do not include the elliptic realization or this comparison.

Proposition 4.6(ii)–(iii) and Remark 7.25 illustrate why duality and a formal Frobenius relation do not supply the positive form automatically. In particular, the formal curve is not the cohomology of a curve.

There is a different established implication over a function field. Let $C/\mathbb{F}_q$ be smooth, projective, and geometrically connected, and let $K = \mathbb{F}_q(C)$. Let $X/\mathbb{F}_q$ be a smooth projective surface with a proper flat morphism $\pi : X \to C$, a zero section, and smooth geometrically connected generic fibre E/K of genus one. Closed fibres of π may be singular. The divisor Tate conjecture for X may be written in pole-order form

$$-\operatorname{ord}_{s=1} \zeta(X, s) = \operatorname{rank} \mathrm{NS}(X),$$

with divisors understood over $\mathbb{F}_q$. Its equivalence with the divisor cycle-map form is an arithmetic theorem [Ulm12, Theorem 6.1.2].

Let $\mathcal{E}/C$ be the Néron model of E, let $t = \#E(K)_{\mathrm{tors}}$, and put $c_v = \#\Phi_v(k_v)$ for its component group at v. Let $R_{\log}$ be the determinant of the logarithmic Néron–Tate height pairing on a basis of $E(K)$ modulo torsion, with $R_{\log} = 1$ in rank zero. In this convention, the full function-field coefficient formula is

$$\lim_{s \to 1} \frac{L(E/K, s)}{(s-1)^r} = \frac{\#\,\mathrm{Sha}(E/K)\, R_{\log} \prod_v c_v}{t^2} q^{\chi(C, \mathrm{Lie}\, \mathcal{E})}. \tag{2.1}$$

The height takes values in $\mathbb{Q} \log q$; a height divided by $\log q$ requires multiplying its rank-r regulator by $(\log q)^r$. The exponent is the coherent Euler characteristic $h^0 - h^1$. This is the elliptic specialization of [LRS22, Conjecture 1.3, (1.10) and §1.4]; its discriminant includes the torsion denominator.

Theorem 2.6 (Tate to function-field BSD)**.** *Let $C/\mathbb{F}_q$ be smooth, projective, and geometrically connected, and let $K = \mathbb{F}_q(C)$. Let $\pi : X \to C$ be a proper flat morphism from a smooth projective surface $X/\mathbb{F}_q$, with a zero section and smooth geometrically connected generic elliptic fibre E/K. If the divisor Tate pole-order conjecture holds for X, then rank BSD, finiteness of* $\mathrm{Sha}(E/K)$*, and* (2.1) *hold for E/K.*

Proof. The Tate–Shioda comparison identifies the defect between the pole order of $\zeta(X, s)$ and $\operatorname{rank} \mathrm{NS}(X)$ with $\operatorname{ord}_{s=1} L(E/K, s) - \operatorname{rank} E(K)$; equivalently, the rank equalities are equivalent [LRS22, Proposition 2.4]. The Artin–Tate–Milne theorem upgrades the Tate pole-order assertion to the Artin–Tate special-value conjecture and finiteness of the Brauer group of X [Ulm12, Theorem 6.1.4]. The Artin–Grothendieck comparison identifies Brauer finiteness with $\mathrm{Sha}(E/K)$ finiteness. Kato–Trihan's characteristic-p theorem then yields the complete BSD statement, including the characteristic-prime part. These imported implications are assembled, under the stated surface hypotheses, in [LRS22, Theorem 1.1 and §1.1]; see also [KT03, p. 541]. □

Thus the divisor Tate conjecture is imposed on the *surface* X, while BSD concerns its generic elliptic curve over K. This implication uses arithmetic comparison theorems in addition to

positivity or Frobenius purity. It is not a deduction of number-field BSD from Grothendieck's standard conjectures. In particular, the pole-order equality is not obtained by matching individual fixed classes with points; the Tate–Shioda comparison subtracts the same vertical and base contributions on both sides.

Proposition 2.7. *Let f be meromorphic near $s = 1$ with nonzero leading coefficient, and let $c > 0$ with $c \neq 1$. Then f and cf have the same divisor but different leading coefficients. The Euclidean line $\mathbb{R}$ has lattices $\mathbb{Z}$ and $2\mathbb{Z}$ with the same real span and metric but Gram determinants 1 and 4.*

Proof. Multiplying by c scales the first nonzero Taylor coefficient without creating a zero or pole. The displayed lattices have bases 1 and 2, whose squared lengths are 1 and 4. □

The first example concerns divisor-only data: a specified Euler product fixes its own normalization. The second shows that a complex metric alone does not name an integral lattice or regulator. Tensoring an arithmetic group with $\mathbb{Q}$ also discards finite torsion.

Proposition 2.8. *Let $G = \bigoplus_{\ell \text{ prime}} \mathbb{Z}/\ell\mathbb{Z}$. For every prime ℓ, the ℓ-primary subgroup $G[\ell^\infty]$ is finite, but G is infinite.*

Proof. The ℓ-primary subgroup is its one summand of order ℓ; there are infinitely many nonzero summands. □

This group example shows why prime-primary finiteness alone is not the whole-group finiteness in BSD. It is not an example of an elliptic curve with infinite Sha.

Alternating cancellation can lose still more graded information, as Proposition 7.30 shows in a Weil datum. For a number-field elliptic curve, the rational period–regulator comparison and the integral comparisons at each prime in the Tamagawa-number conjecture supply additional conjectural input for the full number-field BSD formula; rank equality alone supplies none of those factors [Fla04, §§1–3, Conjectures 1–3 and §5.4]. The elementary examples do not prove that all standard conjectures are logically insufficient for BSD. They identify precisely the missing comparisons in the direct positivity argument.

2.3. **The explicit formula and Weil's criterion.** Weil generalized Riemann's explicit formula to Hecke L-functions and to curves over finite fields, wrote its geometric side as a sum of local terms, and observed that the Riemann hypothesis is equivalent to the positivity of the resulting Hermitian functional on test functions of the form $g * \tilde{g}$ [Wei52], as reported in [Bom06, §V, pp. 8–9]. He later extended the criterion to Artin–Hecke L-functions, where positivity is equivalent to the Riemann hypothesis together with Artin's conjecture, and called the analogy with the function-field case, where positivity follows from the index theorem on $C \times C$, the most serious argument he knew in favour of these conjectures [Wei72, §16, p. 16]. Yoshida and Bombieri studied the functional on test functions of bounded support: positivity holds unconditionally when the support is short enough that no prime power enters, and the Riemann hypothesis is equivalent to nondegeneracy on every support [Yos92, Proposition 1, p. 285, Theorems 1 and 2, pp. 310 and 321], [Bom00, Theorems 1 and 12, pp. 191 and 226–227], [Bur00, Theorem, p. 2]. Connes and Consani gave an operator-theoretic reason for the archimedean positivity in the prime-free window [CC20, Theorem 1, p. 3].

2.4. **Deninger's program.** Deninger showed that Serre's archimedean local factors are zeta-regularized determinants of a derivation on an infinite-dimensional "archimedean cohomology" [Den91, Theorem (4.1), p. 252], and that the local factors at the finite primes are regularized determinants of a derivation whose eigenvalues are all the values ρ with q^ρ a Frobenius eigenvalue [Den92, Proposition (2.3), p. 142, and Theorem (2.7), p. 144]. In §3 of the same paper, labelled

speculative, he proposed an arithmetic site whose cohomology with its flow would give the completed zeta function as an alternating product of regularized determinants, with $\xi_{\mathrm{D}}(s) = \det_\infty((s-\Theta)/2\pi \mid H^1)$ [Den92, (3.1)–(3.2), pp. 146–147], where $\xi_{\mathrm{D}}(s) = 2^{-1/2}(2\pi)^{-2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$ is Deninger's normalization (it differs from ξ of Definition 1.16 by the constant recorded after Theorem 7.32). The resulting prediction $\prod_\rho (s-\rho)/2\pi = \xi_{\mathrm{D}}(s)$ was proved by Deninger for $\operatorname{Re} s > 1$ [Den92, Theorem 3.3, p. 147], extended to the cut plane by Soulé (in a letter reported in [Den92, Remark 3.4(1), p. 147]; see [SS94], as reported in [Den98, p. 168]), and generalized to Hecke L-series of number fields by Illies (Diplomarbeit, Münster 1992, as reported in [Den94, Theorem 7.18, p. 35]); an earlier related formula for every number field, up to a constant and assuming the Riemann hypothesis, is Kurokawa's [Kur88, p. 24]. In Section 7 of [Den94], again explicitly speculative, Deninger described a hypothetical site with an $\mathbb{R}$-action, a Lefschetz trace formula equivalent to Weil's explicit formula, the determinant formula, Poincaré duality, and the argument that a flow-equivariant Hodge star making $\operatorname{Tr}(f \cup *f')$ positive definite would force the eigenvalues onto the critical lines, "by adapting Serre's argument" [Den94, (7.2)–(7.13), pp. 29–33, and (7.19), p. 36]. This is the earliest statement of the implication positivity $\Rightarrow$ Riemann hypothesis among the sources cited here; it is restated in [Den98, pp. 168–169 and p. 172], [Den10, pp. 4–5], and [Den23, 2.7, p. 3]. The same paper already contemplates a Künneth formula for $\operatorname{Spec}\mathbb{Z} \boxtimes \operatorname{Spec}\mathbb{Z}$ and notes that the resulting regularized product over sums of zeros does not make sense [Den94, (7.32), pp. 41–42].

Deninger's ICM address [Den98] rewrites the explicit formula as a distributional Lefschetz formula, conjectures the corresponding trace formula for flows on foliated spaces, and derives the expected shape of "$\operatorname{Spec}\mathbb{Z}$": three-dimensional, with closed orbits of length $\log p$ for the primes and one fixed point for the infinite place [Den98, (5)–(6), pp. 170–171, (20)–(22), pp. 179–180, and p. 181]. The dynamical analogue of the positivity argument is a theorem for Riemannian foliations [Den01, Theorem 2.1, p. 8], and for an ordinary elliptic curve over $\mathbb{F}_q$ the explicit formula is realized as a transversal index formula on a solenoid [Den02]; Deninger and Singhof showed that the leafwise Hodge decomposition needed in general fails for some foliations [DS01, Proposition 1.2, p. 214]. Deninger's note on the Hilbert–Pólya strategy restates the determinant conjecture and the positivity argument and relates the positive form on one eigenspace to the Beilinson–Bloch height pairing [Den10, Conjecture 2, p. 3, pp. 4–5, and pp. 7–9]. Flach and Morin codified the expected properties as a list of axioms, including an antilinear Hodge star with positive form, and related them to unconditionally defined Weil–Arakelov complexes [FM20, Conjecture 2.3, p. 522, Conjecture 4.1, and Theorem 4.3, p. 527]. Finally Deninger proved that no functorial cohomology with real coefficients and an endomorphism with the expected Galois multiplicities can exist for rings of integers, because of symplectic Artin representations of root number -1, crediting the finite-field form of the obstruction to Serre [Den23, §1, p. 1, and Theorem 2.13, p. 5]. None of these works constructs the full global dynamical cohomology with the spectral realization and positive star required here.

2.5. **Products, the absolute point, and the adele class space.** Manin advocated Deninger's formalism together with Kurokawa's tensor product of zeta functions, whose zeros are the sums of zeros, as heuristics for "absolute motives" and Descartes powers of $\operatorname{Spec}\mathbb{Z}$ over an absolute point, and recorded the difficulty that for $\zeta \otimes \zeta$ the zeros ρ and $1-\rho$ make the construction ill-defined [Man95, p. 121, p. 132, and §5.2, p. 158]; Kurokawa's definition restricts to zeros with imaginary parts of equal sign [Kur92, p. 219]. Connes realized the critical zeros as an absorption spectrum on the adele class space and showed, for global fields of positive characteristic, that a global trace formula is equivalent to the Riemann hypothesis for all Grössencharakter L-functions; the global formula is left open [Con99, Theorem 1, p. 13, Theorem 5, p. 42, and p. 3]. Connes and Consani constructed the arithmetic site and the scaling

site, whose points recover the adele class space modulo $\widehat{\mathbb{Z}}^{\times}$ and whose periodic orbits carry a Riemann–Roch formula [CC14a, Theorem 2.6], [CC16, Theorem 1.1, p. 2], [CC15, Theorems 5.1 and 6.7], [CC17, Theorems 3.9 and 5.17], and proposed to derive the Riemann hypothesis from a Riemann–Roch theorem and an intersection theory on the square of the scaling site, which remain to be found [CC19, §3, pp. 16–18, and p. 22], [Con16, §4.3.2, pp. 20–21]; see also the dictionary of [CCM09, §2.4, p. 12] and the one-dimensional Riemann–Roch theorem on $\overline{\operatorname{Spec}\mathbb{Z}}$ of [CC23, Theorem 1.1, p. 2].

2.6. **De Branges spaces.** De Branges proposed to derive the Riemann hypothesis from positivity conditions on the Hilbert space of entire functions with structure function $E(z)=\xi(1-iz)$ and on its reciprocal space [dB86], [dB92], as reported in [CL00, Theorem 2, p. 4]; Conrey and Li showed by high-precision numerical evaluation that these conditions fail, and report a proof by Sarnak, without numerical calculation, for the reciprocal space [CL00, (3.2) and (3.4), pp. 6–7, and Remark, p. 9]. Lagarias proved that $E(z)=\xi(\frac{1}{2}-iz)+\xi'(\frac{1}{2}-iz)$ is a de Branges function if and only if the Riemann hypothesis holds [Lag06a, Theorem 1, p. 9], and that shifted versions are de Branges functions unconditionally [Lag05, Theorem 2.1, p. 8]. Suzuki constructed canonical systems with positive Hamiltonians for shifted ξ unconditionally [Suz12, Theorem 2.3, p. 8], [Suz21, Theorem 2.4, p. 9], and, assuming the Riemann hypothesis, identified the completion of $C_c^{\infty}(\mathbb{R})$ for Weil's form with a model space of a de Branges space [Suz25, Theorem 1.1, p. 2]; a later paper corrects the even extension of the kernels used there [Suz26, p. 27].

2.7. **Surveys and formalizations.** Surveys of this circle of ideas include Bombieri's and Sarnak's problem descriptions [Bom06], [Sar04], Conrey's [Con03], Lagarias's [Lag06b], Connes's [Con00], [Con16], the physically oriented [BK99] and [SH11], Milne's history of the finite-field case [Mil15], and Deninger's own surveys [Den98], [Den01], [Den08].

Formalizations related to these notes include the following.

- The statement of the Riemann hypothesis in Lean, due to D. Loeffler: [TL26] `github.com/leanprover-community/mathlib4`.
- An independent Lean formalization of the Hilbert–Pólya reduction, contemporaneous with the one of Subsection 14.13: [Csi26] `github.com/frostynumeral/hilbert-polya-lean-verification`.
- Li's criterion: [Bul26] `github.com/nicholasbulka/li-criterion-rh-equivalence-lean`.
- Weil's explicit formula for ζ: [Ant26d] `github.com/anthropics/formal-math`.
- The Riemann hypothesis for hyperelliptic curves over finite fields, by the method of Stepanov and Bombieri and without cohomology: [MC26] `github.com/math-inc/RiemannHypothesisCurves`.

The overlaps with the formalization of these notes are listed in Subsection 14.13.

3. Conventions and analytic preliminaries

Throughout, Hermitian forms are linear in the first argument and antilinear in the second. For a meromorphic function $f\not\equiv 0$ on $\mathbb{C}$ and $\rho\in\mathbb{C}$ we write $\operatorname{ord}_\rho f\in\mathbb{Z}$ for its order at ρ; for an entire $f\not\equiv 0$ its *divisor* is the formal sum $\operatorname{div} f=\sum_\rho \operatorname{ord}_\rho(f)[\rho]$.

Definition 1.16 is given in the introduction.

For $K=\mathbb{Q}$ one has $D_{\mathbb{Q}}=1$, $r_1=1$, $r_2=0$, so $\Lambda_{\mathbb{Q}}=\Lambda$ and $\xi_{\mathbb{Q}}=2\xi$. The classical analytic facts used below are collected, with sources, in Section 3.1. By Theorem 3.4, F is an even real entire function of order one whose zeros, counted with multiplicity, correspond to the nontrivial zeros of ζ under $\rho=\frac{1}{2}+i\tau\leftrightarrow z=-\tau$; the Riemann hypothesis (RH) is the statement that all zeros of F are real.

Proposition 3.1. *Let F be as in Definition 1.16. Then $F(0) = \xi(\frac{1}{2}) > 0$.*

Proof. For real $0 < s < 1$ the alternating series $\eta(s) = \sum_{n\geq 1}(-1)^{n-1}n^{-s}$ converges, and grouping its terms in consecutive pairs shows $\eta(s) > 0$. On $\operatorname{Re} s > 1$ one has $\eta(s) = (1 - 2^{1-s})\zeta(s)$, and both sides are analytic on $\operatorname{Re} s > 0$, $s \neq 1$ (the left by Dirichlet's test and locally uniform convergence of the paired series), so the identity persists there. For $0 < s < 1$ we have $1-2^{1-s} < 0$, hence $\zeta(s) < 0$. Since $\Gamma(\frac{1}{4}) > 0$, $\Lambda(\frac{1}{2}) < 0$ and $\xi(\frac{1}{2}) = \frac{1}{2}\cdot\frac{1}{2}\cdot(-\frac{1}{2})\Lambda(\frac{1}{2}) > 0$. □

Definition 3.2. Let $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$.

(a) The *reflection* of a function E defined on a set $\Omega \subset \mathbb{C}$ with $\overline{\Omega} = \Omega$ is the function $E^{\#}(z) = \overline{E(\bar{z})}$ on Ω.
(b) An entire function E satisfies the *Hermite–Biehler (HB) inequality* if $|E^{\#}(z)| < |E(z)|$ for every $z \in \mathbb{C}_+$. (Common real zeros of E and $E^{\#}$ are allowed; when a convention forbids them we say so.)
(c) For an entire function G, its *HB companion* is the entire function $E_G = G + iG'$.
(d) The *Hardy space* $H^2(\mathbb{C}_+)$ is the space of holomorphic G on $\mathbb{C}_+$ with $\sup_{y>0}\int_{\mathbb{R}} |G(x+iy)|^2\,dx < \infty$.

Definition 1.2 is stated in the introduction.

Definition 1.7 is given in the introduction.

The identity (1.1) is Deninger's [Den94, (7.13), p. 33]. Equivalently, $\Theta - j/2$ is skew-symmetric for h. On a finite-dimensional Θ-stable block, (1.1) integrates to $h(\phi^t u, \phi^t v) = e^{jt}h(u,v)$: the flow is a similitude of weight j.

Definition 1.1 is given in the introduction.

On a spectral core the flow of Definition 1.2 is defined blockwise on the generalized eigenspaces. Spectral cores are the algebraic part of the spaces in [FM20, Definition 2.1(i), p. 521] and [Den94, §1]; the condition of no finite accumulation point is added here. The Tate twist of Definition 1.2 is [Den98, p. 168] and [Den10, p. 4].

The following linear algebra is used throughout, from the finite-field calibration on.

Proposition 3.3 (Serre; Deninger [Den94, (7.11)–(7.13), pp. 32–33])**.** *Let V be a complex vector space, $\Theta : V \to V$ a linear map, $j \in \mathbb{R}$, and h a positive infinitesimal polarization of weight j on (V, Θ) (Definition 1.7). Then:*

(i) every eigenvalue ρ of Θ satisfies $\operatorname{Re}\rho = j/2$;
(ii) eigenvectors of Θ with distinct eigenvalues are h-orthogonal;
(iii) Θ is diagonalizable on every finite-dimensional Θ-stable subspace of V; in particular, if (V, Θ) is a spectral core then $V_{\Theta\sim\rho} = \ker(\Theta - \rho)$ for every ρ.

Throughout the proof put $A = \Theta - j/2$; by (1.1), $h(Au, v) = -h(u, Av)$.

Proof of (i). If $\Theta v = \rho v$ with $v \neq 0$, then (1.1) with $u = v$ gives $(\rho + \bar{\rho} - j)h(v,v) = 0$, and $h(v,v) > 0$. □

Proof of (ii). Let $\Theta u = \rho u$ and $\Theta v = \sigma v$ with $\rho \neq \sigma$. By (i), $\rho = j/2 + i\tau$ and $\sigma = j/2 + i\tau'$ with $\tau \neq \tau'$ real. Then (1.1) gives $(\rho + \bar{\sigma} - j)h(u,v) = i(\tau - \tau')h(u,v) = 0$. □

Proof of (iii). Let W be finite-dimensional and Θ-stable. If $\Theta|_W$ were not diagonalizable, some eigenvalue ρ would admit $w \in W$ with $v = (\Theta - \rho)w \neq 0$ and $(\Theta - \rho)v = 0$. By (i), $\rho - j/2 = i\tau$ with $\tau \in \mathbb{R}$, so $\Theta - \rho = A - i\tau$, and $A - i\tau$ is again skew-symmetric because $i\tau$ is purely imaginary. Hence $h(v,v) = h((A - i\tau)w, v) = -h(w, (A - i\tau)v) = 0$, contradicting $v \neq 0$. For a spectral core apply this to $W = V_{\Theta\sim\rho}$. □

Proposition 3.3 excludes Jordan blocks, not repeated eigenvalues: it says nothing about simplicity. It is the linear algebra of Serre's Kähler analogue of the Weil conjectures [Ser60, Théorème 1, pp. 392–393], in the infinitesimal form in which Deninger uses it [Den94, (7.11)–(7.13), pp. 32–33], [Den98, p. 169]; the semisimplicity clause (iii) is [Den23, 2.7, p. 3].

3.1. **Classical inputs.** The following classical results are used without proof. Each is quoted from a source in which its statement was checked; where the source states the result without proving it, this is said.

Theorem 3.4 (Riemann; Hadamard; de la Vallée Poussin; von Mangoldt)**.** *Let ξ and F be as in Definition 1.16. Then:*

(i) ξ extends to an entire function with $\xi(s) = \xi(1-s)$, and all zeros of ξ lie in $0 < \operatorname{Re} s < 1$ [Sut17a, Theorem 17.17];

(ii) ξ is an entire function of order one, and the number $N(T)$ of its zeros ρ with $0 < \operatorname{Im}\rho \le T$, counted with multiplicity, satisfies $N(T) = O(T\log T)$ as $T \to \infty$ [PM18, pp. 12–14], [Rie59, p. 4].

The source for (i) contains a proof; the sources for (ii) state it (in the second, heuristically) with references to the standard proofs. The absence of zeros of ξ on $\operatorname{Re} s = 1$ in (i) is the nonvanishing of ζ on that line (Hadamard, de la Vallée Poussin). Since $\Gamma_{\mathbb{R}}$ has no zeros and its only poles are at $0, -2, -4, \dots$, the zeros of ξ are exactly the zeros of ζ in $0 < \operatorname{Re} s < 1$, with multiplicity: the *nontrivial zeros*. Because ξ is real on $\mathbb{R}$, $\xi(\bar{s}) = \overline{\xi(s)}$. Consequently F is an entire function of order one, even (from $\xi(s) = \xi(1-s)$), and real ($F(\bar{z}) = \overline{F(z)}$); its zeros lie in the strip $|\operatorname{Im} z| < \frac{1}{2}$ and correspond to the nontrivial zeros under $\rho = \frac{1}{2} + i\tau \leftrightarrow z = -\tau$; and the number of zeros of F in $|z| \le R$ is $O(R\log R)$, because such a zero corresponds to a zero ρ with $|\operatorname{Im}\rho| \le R$ and the zeros are symmetric under $\rho \mapsto \bar{\rho}$.

Theorem 3.5 (Hecke)**.** *Let K be a number field and Λ_K as in Definition 1.16. Then Λ_K extends to a meromorphic function on $\mathbb{C}$, holomorphic except for simple poles at $s = 0$ and $s = 1$, and $\Lambda_K(s) = \Lambda_K(1-s)$* [Sut17b, Remark 19.13].

The source states this result, attributing it to Hecke, without proof, and normalizes $\Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\Gamma(s)$; the two normalizations differ by the nonzero constant factor 2^{r_2}, which changes neither the poles nor the functional equation.

Proposition 3.6. *Let K be a number field and ξ_K as in Definition 1.16. Then ξ_K is entire, $\xi_K(s) = \xi_K(1-s)$, every zero of ξ_K lies in the strip $0 \le \operatorname{Re} s \le 1$, $\xi_K(0)\xi_K(1) \neq 0$, and for $0 \le \operatorname{Re} s \le 1$, $s \notin \{0,1\}$, the order of ξ_K at s equals the order of ζ_K at s. In particular every zero of ζ_K with $0 < \operatorname{Re} s < 1$ is a zero of ξ_K of the same order.*

Proof. By Theorem 3.5, $s(s-1)\Lambda_K(s)$ has removable singularities at 0 and 1, so ξ_K is entire, and $\xi_K(1-s) = (1-s)(-s)\Lambda_K(1-s) = \xi_K(s)$. A simple pole has nonzero residue, so $\xi_K(1) = \operatorname{Res}_{s=1}\Lambda_K \neq 0$, and $\xi_K(0) = \xi_K(1)$. For $\operatorname{Re} s > 1$ the Euler product $\zeta_K(s) = \prod_{\mathfrak{p}}(1 - N\mathfrak{p}^{-s})^{-1}$ converges absolutely, so $\zeta_K(s) \neq 0$; the factors $|D_K|^{s/2}$, $\Gamma_{\mathbb{R}}(s)$, $\Gamma_{\mathbb{C}}(s)$ have no zeros; hence $\xi_K(s) \neq 0$ for $\operatorname{Re} s > 1$, and by the functional equation for $\operatorname{Re} s < 0$. For $0 \le \operatorname{Re} s \le 1$ and $s \notin \{0,1\}$ the factors $s(s-1)|D_K|^{s/2}\Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}$ are holomorphic and nonzero (Γ has no zeros and its poles are at $0, -1, -2, \dots$), so ξ_K and ζ_K have the same order there. □

For $K = \mathbb{Q}$ this recovers the correspondence between zeros of $\xi_{\mathbb{Q}} = 2\xi$ and nontrivial zeros of ζ.

Definition 3.7. Let q be a prime power.

(a) The *zeta function* of a variety X over $\mathbb{F}_q$ is the power series
$$Z(X,t)=\exp\Big(\sum_{n\ge 1}\#X(\mathbb{F}_{q^n})\,\frac{t^n}{n}\Big),$$
and we write $\zeta_X(s)=Z(X,q^{-s})$.

(b) For an elliptic curve E over $\mathbb{F}_q$, the *Frobenius roots* of E are the complex numbers α,β (with multiplicity) such that $\det(1-F^*t\mid H^1)=(1-\alpha t)(1-\beta t)$, where $\ell\nmid q$ is a prime, F^* is geometric Frobenius on $H^1(E_{\overline{\mathbb{F}}_q},\mathbb{Q}_\ell)$ (as in Theorem 3.8 below), computed after embedding $\overline{\mathbb{Q}}_\ell$ in $\mathbb{C}$ (by Theorem 3.8 this polynomial has integer coefficients, so α,β depend on neither ℓ nor the embedding).

Theorem 3.8 ([Del74, (1.5.4) and Théorème (1.6), p. 276])**.** *Let X_0 be a smooth projective variety over $\mathbb{F}_q$, X its base change to $\overline{\mathbb{F}}_q$, $\ell\nmid q$ a prime, and F^* the geometric Frobenius acting on $H^i(X,\mathbb{Q}_\ell)$. Then $Z(X_0,t)=\prod_i \det(1-F^*t\mid H^i(X,\mathbb{Q}_\ell))^{(-1)^{i+1}}$; each $\det(1-F^*t\mid H^i(X,\mathbb{Q}_\ell))$ has integer coefficients independent of ℓ; and every complex conjugate of every root α of its reciprocal polynomial satisfies $|\alpha|=q^{i/2}$.*

These are the statements conjectured by Weil [Wei49, pp. 507–508]. For an elliptic curve E over $\mathbb{F}_q$ we also use the standard facts that H^0, H^1, H^2 of $E_{\overline{\mathbb{F}}_q}$ have dimensions $1,2,1$ and that F^* acts on H^0 by 1 and on H^2 by q; they are not part of the statement quoted above. With them, Theorem 3.8 gives

$$Z(E,t)=\frac{(1-\alpha t)(1-\beta t)}{(1-t)(1-qt)},\qquad |\alpha|=|\beta|=q^{1/2}, \tag{3.1}$$

where α,β are the Frobenius roots of E (Definition 3.7).

3.2. **The Weil form.** By Theorem 3.4 and the discussion after it, every zero of F lies in the strip $|\operatorname{Im} z|<\frac12$ and the number $N(R)$ of zeros of F in $|z|\le R$, counted with multiplicity, satisfies $N(R)=O(R\log R)$; by Proposition 3.1, $F(0)\neq 0$. If (r_n) lists the zeros of F with multiplicity, then

$$\sum_n |r_n|^{-2}<\infty. \tag{3.2}$$

Indeed, $\delta=\min_n|r_n|>0$ because the zeros are discrete and $F(0)\neq 0$, and grouping the zeros into the dyadic shells $2^k\delta\le|r|<2^{k+1}\delta$ bounds the sum by $\sum_{k\ge0}N(2^{k+1}\delta)(2^k\delta)^{-2}$, which converges since $N(R)=O(R\log R)$.

Definition 3.9. Let F be as in Definition 1.16, let Γ be the set of distinct zeros of F, and for $\gamma\in\Gamma$ let $m_\gamma=\operatorname{ord}_\gamma F\ge 1$.

(a) The *Fourier transform* of a compactly supported $\psi\in L^1(\mathbb{R})$ is the entire function $\widehat{\psi}(z)=\int_{\mathbb{R}}\psi(t)e^{izt}\,dt$, $z\in\mathbb{C}$.

(b) The *adjoint* of a function ψ on $\mathbb{R}$ is $\widetilde{\psi}(t)=\overline{\psi(-t)}$.

(c) The *Weil distribution* is the linear functional $W(f)=\sum_{\gamma\in\Gamma}m_\gamma\widehat{f}(\gamma)$ on $C_c^\infty(\mathbb{R})$; the series converges absolutely by Proposition 3.10.

(d) The *Weil form* is $Q_W(\psi)=W(\psi*\widetilde{\psi})$ for $\psi\in C_c^\infty(\mathbb{R})$.

The Fourier inversion formula for this convention carries the factor $1/2\pi$. The sign of the exponent does not affect any statement about the location of zeros, because F is even. The form Q_W is Weil's quadratic functional [Wei72, §16, p. 16]; see [Bom06, §V, p. 9] and [Bom00, Theorem 1, p. 191].

Proposition 3.10. *Let F, Γ, m_γ, W, and Q_W be as in Definition 3.9, and let $f, \psi \in C_c^\infty(\mathbb{R})$. Then:*

(i) $\psi * \widetilde{\psi} \in C_c^\infty(\mathbb{R})$ and $(\psi * \widetilde{\psi})\widehat{\ }(z) = \widehat{\psi}(z)\,\overline{\widehat{\psi}(\bar{z})}$ for all $z \in \mathbb{C}$;

(ii) the series $\sum_\gamma m_\gamma \widehat{f}(\gamma)$ converges absolutely, and

$$Q_W(\psi) = \sum_{\gamma \in \Gamma} m_\gamma\, \widehat{\psi}(\gamma)\, \overline{\widehat{\psi}(\bar{\gamma})},$$

an absolutely convergent series whose sum is real;

(iii) if every zero of F is real, then $Q_W(\psi) = \sum_\gamma m_\gamma |\widehat{\psi}(\gamma)|^2 \geq 0$.

Proof of (i). Convolution of compactly supported smooth functions is compactly supported and smooth. Since $e^{iz(s+u)} = e^{izs}e^{izu}$, Fubini's theorem gives $(g * h)\widehat{\ } = \widehat{g}\,\widehat{h}$ for compactly supported integrable g, h. Moreover, substituting $u = -t$,

$$\widehat{\widetilde{\psi}}(z) = \int_{\mathbb{R}} \overline{\psi(u)} e^{-izu}\, du = \overline{\int_{\mathbb{R}} \psi(u) e^{i\bar{z}u}\, du} = \overline{\widehat{\psi}(\bar{z})},$$

because $\overline{e^{-izu}} = e^{i\bar{z}u}$ for real u. □

Proof of (ii). Let $\operatorname{supp} f \subset [-a, a]$. For $|\operatorname{Im} z| \leq \frac{1}{2}$ and $t \in [-a, a]$ one has $|e^{izt}| \leq e^{a/2}$. Integrating by parts k times, $(-iz)^k \widehat{f}(z) = \widehat{f^{(k)}}(z)$, so $|z|^k |\widehat{f}(z)| \leq e^{a/2} \|f^{(k)}\|_1$. Using $k = 0$ and $k = 2$ and $(1 + |z|)^2 \leq 2(1 + |z|^2)$,

$$|\widehat{f}(z)| \leq C_f (1 + |z|)^{-2} \quad (|\operatorname{Im} z| \leq \tfrac{1}{2}), \qquad C_f = 2e^{a/2}\big(\|f\|_1 + \|f''\|_1\big).$$

All zeros satisfy $|\operatorname{Im}\gamma| < \frac{1}{2}$, so $\sum_\gamma m_\gamma |\widehat{f}(\gamma)| \leq C_f \sum_n |r_n|^{-2} < \infty$ by (3.2). Applying this to $f = \psi * \widetilde{\psi}$ and using (i) gives the displayed formula. Since F is real, $\gamma \mapsto \bar{\gamma}$ permutes Γ with $m_{\bar{\gamma}} = m_\gamma$, and the term indexed by $\bar{\gamma}$ is the complex conjugate of the term indexed by γ; by absolute convergence the sum is real. □

Proof of (iii). If γ is real then $\bar{\gamma} = \gamma$ and the γ-term is $m_\gamma |\widehat{\psi}(\gamma)|^2$. □

4. Finite-field calibration

This section isolates the linear algebra by which positivity forces the Riemann hypothesis for varieties over $\mathbb{F}_q$, in the two forms in which it occurs: a positive form on the cohomology for which Frobenius is a similitude, and a positive involution on an algebra of operators containing Frobenius. Weil's theorem for curves is an instance of the second form, not of the first. The section then shows how either form becomes, after logarithmic suspension of Frobenius, exactly the hypothesis of the purity statement used over number fields (Proposition 3.3). Section 7.5 places the result in the frame of the axioms of Section 5. In applications q is a prime power; the linear algebra only uses that $q > 1$ is real.

It is natural to expect that a finite-dimensional graded cohomology with a Frobenius, perfect Poincaré duality, a rational zeta function with functional equation, and integral point counts already forces $|\alpha| = q^{j/2}$, and that any nondegenerate Frobenius-compatible form does; in fact none of this suffices, and positivity is the whole content (Propositions 4.6 and 4.3).

4.1. **Frobenius polarizations.**

Definition 4.1. Let $q > 1$ be real, $j \in \mathbb{R}$, V a finite-dimensional complex vector space, and $\mathrm{F} : V \to V$ a linear map. A *positive Frobenius polarization of weight j* on (V, F) is a positive definite Hermitian form h on V with $h(\mathrm{F}u, \mathrm{F}v) = q^j h(u, v)$ for all $u, v \in V$.

This is the vector-space form of the hypothesis of Serre's argument [Ser60, Théorème 1 and its proof, pp. 392–393].

Proposition 4.2 (Serre [Ser60, p. 393]; Kleiman [Kle68, Lemma 4.6, p. 385])**.** *Let $q > 1$ be real, $j \in \mathbb{R}$, V a finite-dimensional complex vector space, $\mathrm{F} : V \to V$ linear, and h a positive Frobenius polarization of weight j on (V, F). Then F is diagonalizable in an h-orthonormal basis, distinct eigenspaces of F are h-orthogonal, and every eigenvalue α of F satisfies $|\alpha| = q^{j/2}$.*

Proof. Put $U = q^{-j/2}\mathrm{F}$. Then $h(Uu, Uv) = h(u, v)$, so U is an isometry of the finite-dimensional inner-product space (V, h), hence injective and therefore unitary. A unitary operator on a finite-dimensional inner-product space is diagonalizable in an orthonormal basis with eigenvalues of modulus one (spectral theorem for normal operators), and its eigenspaces are the spans of the basis vectors with a given eigenvalue, hence mutually orthogonal. Hence $\mathrm{F} = q^{j/2}U$ is diagonalizable in the same basis, with the same eigenspaces, and its eigenvalues have modulus $q^{j/2}$. □

Proposition 4.3. *Let $q > 1$ be real, $j \in \mathbb{R}$, V a finite-dimensional complex vector space, and $\mathrm{F} : V \to V$ linear.*

(i) (V, F) admits a positive Frobenius polarization of weight j if and only if F is diagonalizable and all its eigenvalues have modulus $q^{j/2}$.

(ii) In that case the positive Frobenius polarizations of weight j are exactly the orthogonal sums $\bigoplus_\alpha h_\alpha$ of arbitrary positive definite Hermitian forms h_α on the eigenspaces V_α of F.

Proof of (i). One direction is Proposition 4.2. Conversely, the form described in (ii) is a positive Frobenius polarization: for $u \in V_\alpha$, $v \in V_\beta$, $h(\mathrm{F}u, \mathrm{F}v) = \alpha\bar{\beta}h(u, v)$, which is $0 = q^j \cdot 0$ if $\alpha \neq \beta$ and $|\alpha|^2 h(u, v) = q^j h(u, v)$ if $\alpha = \beta$. □

Proof of (ii). The computation just made shows that every such orthogonal sum is a positive Frobenius polarization. Conversely, a positive Frobenius polarization makes distinct eigenspaces orthogonal (Proposition 4.2), so it is the orthogonal sum of its restrictions to the eigenspaces. □

On a single finite-dimensional space, the existence of a positive Frobenius polarization is therefore *equivalent* to purity together with semisimplicity, and when one exists it is far from unique. The content of the positivity hypothesis lies entirely in producing a form from geometry; the same phenomenon over $\operatorname{Spec}\mathbb{Z}$ is Proposition 7.7.

It is natural to expect that purity of the eigenvalues is all a positive Frobenius polarization encodes; in fact it also encodes semisimplicity, and purity alone does not produce one.

Proposition 4.4. *Let $q > 1$ be real, $j \in \mathbb{R}$, $V = \mathbb{C}^2$, and $\mathrm{F} = \begin{pmatrix} q^{j/2} & 1 \\ 0 & q^{j/2} \end{pmatrix}$. Then every eigenvalue of F has absolute value $q^{j/2}$, and (V, F) admits no positive Frobenius polarization of weight j.*

Proof. The only eigenvalue is $q^{j/2}$, and F is not diagonalizable; Proposition 4.2 excludes a positive Frobenius polarization. □

For a nonsemisimple Frobenius one must therefore specify separately how its nilpotent part is transported to a generator (Definition 4.15); a positive realization does not follow from absolute values.

Definition 4.5. The *formal curve* is the graded $\mathbb{C}$-algebra $H^\bullet$ with $H^0 = \mathbb{C} \cdot 1$, $H^1 = \mathbb{C}e_1 \oplus \mathbb{C}e_2$, $H^2 = \mathbb{C}\eta$, unit 1, products $e_1 \cup e_2 = -e_2 \cup e_1 = \eta$ and all other products of positive-degree

elements zero, trace $\mathrm{Tr}(c\eta) = c$, and the graded linear map F equal to 1 on H^0, to $\mathrm{diag}(3,1)$ on H^1 in the basis e_1, e_2, and to 3 on H^2; put $q = 3$. Its *Lefschetz numbers* are $N_n = \sum_i (-1)^i \,\mathrm{Tr}(\mathrm{F}^n \mid H^i)$ for $n \geq 1$, and its *zeta function* is $Z(t) = \prod_i \det(1 - \mathrm{F}t \mid H^i)^{(-1)^{i+1}}$.

Proposition 4.6. *Let $H^\bullet$, F, $q = 3$, N_n, and $Z(t)$ be as in Definition 4.5.*

(i) F *is an automorphism of the graded algebra; the pairing $B(u,v) = \mathrm{Tr}(u \cup v)$ on H^1 is alternating and perfect with $B(\mathrm{F}u, \mathrm{F}v) = qB(u,v)$; $N_n = 0 \in \mathbb{Z}$ for all n; $Z(t) = P_1(t)/((1-t)(1-qt))$ with $P_1(t) = (1-t)(1-3t) \in \mathbb{Z}[t]$, and $q\,t^2 P_1(1/(qt)) = P_1(t)$.*

(ii) The eigenvalues 3 and 1 of F *on H^1 do not have modulus $q^{1/2}$.*

(iii) The Hermitian forms h on H^1 with $h(\mathrm{F}u, \mathrm{F}v) = q\,h(u,v)$ are exactly those with $h(e_1,e_1) = h(e_2,e_2) = 0$; the nonzero ones are nondegenerate of signature $(1,1)$, and none is positive definite.

(iv) On the subalgebra $H^0 \oplus H^2$, which contains the classes of the unit and of a point, the pairing $(a \cdot 1 + b\eta, a' \cdot 1 + b'\eta) \mapsto ab' + ba'$ is nondegenerate and F*-compatible of weight one, and* F *is semisimple with eigenvalues 1 and q there.*

Proof of (i). $\mathrm{F}(e_1 \cup e_2) = 3\eta = \mathrm{F}e_1 \cup \mathrm{F}e_2$, and the other products are compatible trivially. $B(e_1,e_2) = 1 = -B(e_2,e_1)$ and $B(e_i,e_i) = 0$, so B is alternating and perfect, and $B(\mathrm{F}e_1, \mathrm{F}e_2) = 3 = qB(e_1,e_2)$. Next $N_n = 1 - (3^n + 1) + 3^n = 0$. The formula for Z is the definition, and $3t^2 P_1(1/(3t)) = 3t^2(1 - \frac{1}{3t})(1 - \frac{1}{t}) = (3t-1)(t-1) = P_1(t)$. □

Proof of (ii). $|3| = 3$ and $|1| = 1$, while $q^{1/2} = \sqrt{3}$. □

Proof of (iii). Write $h_{kl} = h(e_k, e_l)$. The condition reads $f_k f_l h_{kl} = 3h_{kl}$ with $(f_1, f_2) = (3,1)$, that is $9h_{11} = 3h_{11}$, $h_{22} = 3h_{22}$, and $3h_{12} = 3h_{12}$. So $h_{11} = h_{22} = 0$ and $h_{12} = \overline{h_{21}}$ is arbitrary; the Gram matrix $\left(\begin{smallmatrix} 0 & c \\ \bar{c} & 0 \end{smallmatrix}\right)$ has eigenvalues $\pm|c|$. □

Proof of (iv). The pairing is $(u,v) \mapsto \mathrm{Tr}(u \cup v)$ restricted to $H^0 \oplus H^2$ (with Tr extended by zero on H^0), its Gram matrix in the basis $1, \eta$ is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and $\mathrm{Tr}(\mathrm{F}u \cup \mathrm{F}v) = q\,\mathrm{Tr}(u \cup v)$ because F is multiplicative and is q on H^2. □

The formal curve is linear algebra, not the cohomology of a curve: $N_n = 0$ for all n is impossible for a curve over $\mathbb{F}_3$ by the Hasse–Weil bound itself. Item (iv) shows that nondegeneracy and Frobenius compatibility of the intersection pairing on the algebraic classes of X itself (for a curve, the classes of degree 0 and 2) say nothing about H^1; since F has eigenvalues 1 and q there, no Frobenius-compatible pairing on these classes can be positive, so positivity has to be sought elsewhere. The positivity used in Weil's proof lives instead on the algebra of correspondences (Section 4.3), and in Grothendieck's formulation it is the Hodge standard conjecture on $X \times X$ together with the Lefschetz-type standard conjecture [Kle68, Introduction, pp. 359–360], [Mil15, Definition 3.6, p. 39, and Theorems 3.15 and 3.17, p. 42]. The finite-field form of the quaternionic block of Proposition 6.11, with $\mathrm{F} = q^{\Theta}$, is Proposition 9.1.

4.2. **The algebra form of positivity.**

Definition 4.7. Let $k \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$ and A a finite-dimensional unital k-algebra. A *positive trace involution* on A is a pair $(\tau, \dagger)$ of a k-linear functional $\tau : A \to k$ with $\tau(ab) = \tau(ba)$ and a map $a \mapsto a^\dagger$ with $(ab)^\dagger = b^\dagger a^\dagger$, $a^{\dagger\dagger} = a$, and $\tau(aa^\dagger) > 0$ for $a \neq 0$ (a real number), where $\dagger$ is k-linear if $k \neq \mathbb{C}$; if $k = \mathbb{C}$, $\dagger$ is antilinear and moreover $\tau(a^\dagger) = \overline{\tau(a)}$.

For $k = \mathbb{C}$ it enters the arithmetic axioms (Definition 5.5). For the algebra of correspondences with the transpose and its trace, positivity is [Kle68, 3.11 and Lemma 4.6].

Proposition 4.8 (Kleiman [Kle68, Lemma 4.6, p. 385])**.** *Let $k \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$, A a finite-dimensional unital k-algebra with a positive trace involution $(\tau, \dagger)$, $q > 1$ real, $j \in \mathbb{R}$, and $g \in A$ with $gg^\dagger = q^j$.*

(i) The minimal polynomial of g over k is separable, and each of its complex roots has modulus $q^{j/2}$.

(ii) If V is a finite-dimensional complex vector space and $\iota : A \otimes_k \mathbb{C} \to \mathrm{End}_{\mathbb{C}}(V)$ an injective unital $\mathbb{C}$-algebra homomorphism, then $\iota(g)$ is diagonalizable and all its eigenvalues have modulus $q^{j/2}$.

Proof of (i). In the finite-dimensional algebra A, $gg^\dagger = q^j$ makes g invertible with inverse $q^{-j}g^\dagger$, so also $g^\dagger g = q^j$. We construct a positive definite Hermitian form h on $A_{\mathbb{C}} = A \otimes_k \mathbb{C}$ with $h(ga, gb) = q^j h(a, b)$, where g acts by left multiplication L_g. If $k = \mathbb{C}$, put $h(a, b) = \tau(ab^\dagger)$: it is linear in a, antilinear in b, Hermitian because $\overline{h(b, a)} = \tau((ba^\dagger)^\dagger) = \tau(ab^\dagger)$, positive definite by hypothesis, and $h(ga, gb) = \tau(gab^\dagger g^\dagger) = \tau(ab^\dagger g^\dagger g) = q^j h(a, b)$. If $k \neq \mathbb{C}$, let $S(a, b) = \frac{1}{2}(\tau(ab^\dagger) + \tau(ba^\dagger))$ on $A_{\mathbb{R}} = A \otimes_k \mathbb{R}$. It is symmetric, satisfies $S(ga, gb) = q^j S(a, b)$ by the same computation, and is positive definite: its Gram matrix in a k-basis of A has entries in k and is positive on nonzero vectors with coordinates in k, hence positive semidefinite over $\mathbb{R}$; a real null vector would lie in the kernel of this matrix, which is spanned by vectors with coordinates in k, and there are none. Let h be the Hermitian extension of S to $A_{\mathbb{C}} = A_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. In either case, Proposition 4.2 applied to $(A_{\mathbb{C}}, L_g, h)$ shows that L_g is diagonalizable with eigenvalues of modulus $q^{j/2}$, so its minimal polynomial is separable with roots of that modulus. Since A is unital, $p(L_g) = 0$ if and only if $p(g) = 0$ (evaluate at 1, and conversely $p(L_g)a = p(g)a$), for polynomials p over $\mathbb{C}$; and the minimal polynomial of g over $\mathbb{C}$ in $A_{\mathbb{C}}$ equals its minimal polynomial over k in A. □

Proof of (ii). Since ι is an injective unital algebra homomorphism, $p(\iota(g)) = 0$ if and only if $p(g) = 0$ in $A_{\mathbb{C}}$, so $\iota(g)$ has the minimal polynomial of g, which by (i) is separable with roots of modulus $q^{j/2}$. □

Proposition 4.8 is Kleiman's form of the positivity argument [Kle68, Lemma 4.6, p. 385], which is stated for the normalized Frobenius with $g' \circ g = \mathrm{id}$ and includes the separability of (i); the form $gg^\dagger = q^j$ is [Mil15, Proposition 3.10]. With g replaced by a generator θ and $gg^\dagger = q^j$ by $\theta + \theta^\dagger = j$, the same argument (with Proposition 3.3 in place of Proposition 4.2) gives Proposition 6.2(i), which in this additive form is Deninger's identity [Den94, (7.13), p. 33]; the two are related by $g = q^\theta$ (Proposition 4.17).

4.3. **Weil's theorem for curves.**

Theorem 4.9 (Grothendieck; Deligne)**.** *Let X be a smooth projective geometrically connected variety of dimension d over $\mathbb{F}_q$, $\ell \nmid q$ a prime, $H^j = H^j(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$, and F the geometric Frobenius. Then H^j is finite-dimensional and vanishes outside $0 \le j \le 2d$; the cup product makes $H^\bullet$ an associative graded-commutative algebra with unit, and F is an algebra automorphism; $H^0 = \mathbb{Q}_\ell \cdot 1$ with $\mathrm{F} = 1$, and H^{2d} is one-dimensional with $\mathrm{F} = q^d$; there is a linear isomorphism $\mathrm{tr} : H^{2d} \to \mathbb{Q}_\ell$, and the pairings $(u, v) \mapsto \mathrm{tr}(u \cup v)$ on $H^j \times H^{2d-j}$ are perfect* [Kle94, §3, pp. 10–11], [Mil13, Theorem 24.1, p. 144, §25, p. 147, and proof of Theorem 27.12, p. 160] *(the duality theorem there is stated for torsion coefficients, from which the statement for $\mathbb{Q}_\ell$ follows by passage to the limit). If X is a curve, F acts semisimply on H^1 (Weil;* [Kle94, p. 10]*).*

Here F is the pullback by the q-Frobenius endomorphism, an algebra endomorphism by functoriality, and it is injective because $\mathrm{tr}(\mathrm{F}u \cup \mathrm{F}v) = q^d \,\mathrm{tr}(u \cup v)$ and the pairings are perfect; the trace is identified with $\mathbb{Q}_\ell$ after choosing a generator of $\mathbb{Z}_\ell(1)$, which does not affect anything below.

Theorem 4.10 (Weil; [Mil08])**.** *Let C be a smooth projective geometrically connected curve of genus $g \geq 1$ over $\mathbb{F}_q$, J its Jacobian, $\pi \in \mathrm{End}(J)$ the q-Frobenius endomorphism, λ the canonical principal polarization, $\mathrm{End}^0(J) = \mathrm{End}(J) \otimes \mathbb{Q}$, $\alpha \mapsto \alpha^\dagger = \lambda^{-1}\alpha^\vee\lambda$ the Rosati involution, $\ell \nmid q$ a prime, and $V_\ell J$ the rational ℓ-adic Tate module. Then:*

- *(i) for $\alpha \in \mathrm{End}(J)$ the characteristic polynomial P_α of α on $V_\ell J$ has integer coefficients independent of ℓ, and the map $\mathrm{End}(J) \otimes \mathbb{Q}_\ell \to \mathrm{End}(V_\ell J)$ is injective* [Mil08, Theorem 10.9, p. 46, Theorem 10.15, p. 49, and Proposition 10.20, p. 50]*;*
- *(ii) the bilinear form $(\alpha, \beta) \mapsto \mathrm{Tr}(\alpha \circ \beta^\dagger)$ on $\mathrm{End}^0(J)$, with Tr the trace on $V_\ell J$, is positive definite* [Mil08, Theorem 14.3, p. 62]*;*
- *(iii) $\pi\pi^\dagger = q$* [Mil08, Chapter II, Lemma 1.2, p. 76]*;*
- *(iv) $\#C(\mathbb{F}_q) = 1 - \sum_i a_i + q$, where $a_1, \dots, a_{2g}$ are the roots of P_π* [Mil08, Chapter III, Theorem 11.1, p. 118].

The source proves these statements except that it defers the calculation behind (ii) to other work; a proof of the positivity by intersection theory is also given in [Mil15, Theorem 1.27, p. 21]. The positivity (ii) is due to Weil [Wei48b], as reported in [Mil15, p. 23].

Proposition 4.11 (Weil)**.** *Let C, $g \geq 1$, J, π, and ℓ be as in Theorem 4.10, and let $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ be a field isomorphism. Then π acts semisimply on $V_\ell J \otimes_{\mathbb{Q}_\ell, \iota} \mathbb{C}$, every root a of P_π satisfies $|\gamma(a)| = q^{1/2}$ for every field embedding $\gamma : \mathbb{Q}(a) \to \mathbb{C}$, and $|\#C(\mathbb{F}_{q^n}) - q^n - 1| \leq 2g\, q^{n/2}$ for all $n \geq 1$.*

Proof. By Theorem 4.10(i), $\tau(\alpha) = \mathrm{Tr}(\alpha \mid V_\ell J)$ is a $\mathbb{Q}$-valued linear functional on $A = \mathrm{End}^0(J)$ with $\tau(ab) = \tau(ba)$, and by (ii) it forms with the Rosati involution a positive trace involution over $k = \mathbb{Q}$ (Definition 4.7). By (iii), $\pi\pi^\dagger = q$. By Proposition 4.8(i) with $j = 1$, the minimal polynomial m of π over $\mathbb{Q}$ is separable with all complex roots of modulus $q^{1/2}$. Since $\mathrm{End}^0(J)$ acts on $V_\ell J$ through an algebra homomorphism, $m(\pi) = 0$ on $V_\ell J$, so π is semisimple there (its minimal polynomial divides the separable m), and every root of P_π is a root of m; the conjugates of a root of m are roots of m. For the last assertion apply (iv) to C over $\mathbb{F}_{q^n}$, whose Frobenius endomorphism is π^n with characteristic polynomial roots a_i^n on $V_\ell J$. □

The same instance can be run on the surface $C \times C$. Positivity of the trace form $X \mapsto \mathrm{Tr}(X \circ X')$ on correspondences modulo the trivial ones expresses the positivity of the Rosati involution, the transpose X' corresponding to † and the graph of Frobenius to π [Mil15, fn. 7, p. 5]; on $C \times C$ it follows from the Hodge index theorem [Har77, Theorem V.1.9, p. 364], and [Har77, Exercises V.1.9 and V.1.10, p. 368] derive from it the inequality $|N - q - 1| \leq 2g\sqrt{q}$ directly. Both routes prove $\tau(aa^\dagger) > 0$ for the same algebra and then apply Proposition 4.8; they differ only in how positivity is proved.

It is natural to expect that Weil's positivity is a canonical positive Frobenius polarization on $H^1(C)$, so that Weil's theorem is an instance of Proposition 4.2; in fact the positive form that Weil constructs lives on the endomorphism algebra (Proposition 4.11), the complex space $H^1(C) \otimes_\iota \mathbb{C}$ carries no complex conjugation induced by its ℓ-adic structure, and the Frobenius polarizations on it form the family of Proposition 4.3(ii), none of which is singled out.

Proposition 4.12. *Let C be a smooth projective geometrically connected curve of genus $g \geq 1$ over $\mathbb{F}_q$, $\ell \nmid q$ a prime, $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ a field isomorphism, $H = H^1(C_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$, and $V = H \otimes_{\mathbb{Q}_\ell, \iota} \mathbb{C}$ with geometric Frobenius F.*

- *(i) The rule $v \otimes z \mapsto v \otimes \bar{z}$ does not define a map $V \to V$.*
- *(ii) (V, F) admits positive Frobenius polarizations of weight one, and they are exactly the orthogonal sums of arbitrary positive definite forms on the eigenspaces of F.*

Proof of (i). By Hensel's lemma $\mathbb{Q}_\ell$ contains a square root x of $-m$ for some positive integer m (take $m = 7$ if $\ell = 2$, and m with $-m$ a nonzero square modulo ℓ otherwise). Since ι fixes $\mathbb{Q}$, $\iota(x)^2 = -m$, so $\iota(x)$ is purely imaginary and nonzero. For $v \neq 0$ in H, the vector $xv\otimes 1 = v\otimes\iota(x)$ would be sent by the rule both to $xv\otimes 1$ and to $v\otimes\overline{\iota(x)} = -v\otimes\iota(x) = -(xv\otimes 1)$, which differ. □

Proof of (ii). By Theorem 4.9, F is semisimple on V, and by Theorem 3.8 its eigenvalues have modulus $q^{1/2}$. Apply Proposition 4.3. □

Thus the positivity that Weil proves is an instance of the algebra form (Proposition 4.8), with $A = \mathrm{End}^0(J)$, the Rosati involution, and $g = \pi$: a multiplicative relation $\pi\pi^\dagger = q$, which Section 4.5 recasts over $\mathbb{C}$ as the multiplicative algebra form AW4-$\mathrm{E_F}$ (Proposition 4.20) and converts into the additive relation of the arithmetic axioms (Proposition 4.19). A polarization in the vector-space form on H^1 is obtained only afterwards, from semisimplicity and purity, and is not canonical. The passage from $V_\ell J$ to H^1 goes through the identification of $H^1(C_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ with the dual of $\mathbb{Q}_\ell \otimes T_\ell J$ [Mil15, Example 3.8, p. 40]; its compatibility with Frobenius is not re-derived here. The proof of (ii) uses the imported semisimplicity and purity statements; the proof of Theorem 7.24(vi) uses Proposition 4.20 and this identification instead. The arithmetic axioms carry both the vector-space and the algebra forms of positivity (Definition 5.5).

Remark 4.13. Let X be a smooth projective variety over $\mathbb{F}_q$, $\ell \nmid q$ a prime, and $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ a field isomorphism. By Theorem 3.8, the eigenvalues of geometric Frobenius on $H^j(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell,\iota} \mathbb{C}$ all have absolute value $q^{j/2}$; Deligne's proof does not produce a positive Frobenius polarization. By Proposition 4.3, such a polarization exists if and only if Frobenius also acts semisimply there, and semisimplicity of Frobenius on H^j is known for curves and abelian varieties (Weil) and a few other varieties, is implied by the standard conjectures, and is unknown in general [Kle94, p. 10], [Mil15, Aside 3.18]. For curves, Weil's positivity of the Rosati involution gives semisimplicity and purity of π on $V_\ell J$ (Proposition 4.11); via the identification of H^1 with the dual of the Tate module ([Mil15, Example 3.8, p. 40], Frobenius compatibility not re-derived here) this yields semisimplicity and purity of Frobenius on H^1, and Proposition 4.3 then supplies the non-canonical polarizations of Proposition 4.12.

4.4. **Logarithmic suspension.**

Definition 4.14. Let $q > 1$ be real and $\alpha \in \mathbb{C}^\times$. The *logarithmic suspension* of α is the set $\Sigma_q(\alpha) = \{\rho \in \mathbb{C} : q^\rho = \alpha\}$, where $q^\rho = e^{\rho \log q}$ with $\log q \in \mathbb{R}$.

Explicitly, if ℓ is any complex logarithm of α, then $\Sigma_q(\alpha) = \{(\ell + 2\pi i n)/\log q : n \in \mathbb{Z}\}$; changing ℓ reindexes this set. The set $\Sigma_q(\alpha)$ is the spectrum of Deninger's construction [Den94, Corollary (2.6), p. 8, and (2.8), p. 10]; see also [Hes18, Remark 7.2].

Definition 4.15. Let $q > 1$ be real, V a finite-dimensional complex vector space, and $\mathrm{F} : V \to V$ invertible, with generalized eigenspaces $V_{(\alpha)}$. On $V_{(\alpha)}$ let $N_\alpha = \log(\alpha^{-1}\mathrm{F}) = \sum_{k\geq 1}(-1)^{k+1}(\alpha^{-1}\mathrm{F} - 1)^k/k$, a nilpotent map (the sum is finite). The *logarithmic suspension* of (V, F) is the pair $(\widehat{V}, \Theta)$ with

$$\widehat{V} = \bigoplus_{\rho\in\mathbb{C},\ q^\rho\in\operatorname{spec}\mathrm{F}} \widehat{V}_\rho, \qquad \widehat{V}_\rho = V_{(q^\rho)}, \qquad \Theta|_{\widehat{V}_\rho} = \rho + \tfrac{1}{\log q} N_{q^\rho}$$

(algebraic direct sum, one copy of $V_{(q^\rho)}$ for each ρ).

(a) If h is a Hermitian form on V, the *suspended form* $\widehat{h}$ is the orthogonal sum of the restrictions of h to the summands $\widehat{V}_\rho$.

(b) If $\widehat{h}$ is a Hermitian form on $\widehat{V}$ and (ρ_α) is a choice of one $\rho_\alpha \in \Sigma_q(\alpha)$ for each eigenvalue α of F, the *restricted form* on $V = \bigoplus_\alpha V_{(\alpha)}$ is the orthogonal sum of the restrictions of $\widehat{h}$ to the summands $\widehat{V}_{\rho_\alpha} = V_{(\alpha)}$.

The index set is the union of the sets $\Sigma_q(\alpha)$ over the eigenvalues α of F; no branch of the logarithm is chosen. On each summand $q^\Theta = q^\rho \exp(N_{q^\rho}) = \mathrm{F}$. Definition 4.15 is Deninger's functor from regular singular connections on $\mathbb{G}_m$ with monodromy F [Den94, §2, pp. 6–8], applied to ℓ-adic cohomology as in [Den94, §4, p. 16], written out explicitly; the name is the one used in these notes.

Proposition 4.16 ((i)–(ii) Deninger [Den94, Corollary (2.8), p. 10, and (4.1), p. 16])**.** *Let $q > 1$ be real, V a finite-dimensional complex vector space, $\mathrm{F} : V \to V$ invertible, and $(\widehat{V}, \Theta)$ its logarithmic suspension.*

(i) for each eigenvalue α, the zeros of the entire function $s \mapsto 1 - \alpha q^{-s}$ are exactly the points of $\Sigma_q(\alpha)$, each simple; consequently the zeros of $\det(1 - \mathrm{F}q^{-s})$, counted with multiplicity, are exactly the eigenvalues of Θ on $\widehat{V}$, counted with algebraic multiplicity;

(ii) $(\widehat{V}, \Theta)$ is a spectral core (Definition 1.1) with $\widehat{V}_{\Theta \sim \rho} = \widehat{V}_\rho$;

(iii) if h is a positive Frobenius polarization of weight $j \in \mathbb{R}$ on (V, F), then $\widehat{h}$ is a positive infinitesimal polarization of weight j on $(\widehat{V}, \Theta)$ (Definition 1.7);

(iv) in that case every eigenvalue of Θ has real part $j/2$ and Θ is diagonalizable.

Proof of (i). $1 - \alpha q^{-s} = 0$ if and only if $q^s = \alpha$, that is, $s \in \Sigma_q(\alpha)$. The derivative $\alpha(\log q) q^{-s}$ never vanishes, so each zero is simple. Since $\det(1 - \mathrm{F}q^{-s}) = \prod_\alpha (1 - \alpha q^{-s})^{\dim V_{(\alpha)}}$, its zero at ρ has order $\dim V_{(q^\rho)}$, which is the algebraic multiplicity of ρ as an eigenvalue of Θ, because $\Theta - \rho$ is nilpotent on $\widehat{V}_\rho$ and Θ has no other eigenvalue ρ on the other summands. □

Proof of (ii). On $\widehat{V}_\rho$, $\Theta - \rho = N_{q^\rho}/\log q$ is nilpotent, so $\widehat{V}_\rho$ is the generalized ρ-eigenspace, of finite dimension. The set of eigenvalues is a finite union of arithmetic progressions $\Sigma_q(\alpha)$ with nonzero step $2\pi i/\log q$, so it has no finite accumulation point. □

Proof of (iv) and (iii). By Proposition 4.2, F is diagonalizable with $|\alpha| = q^{j/2}$, so every $N_\alpha = 0$, $\Theta = \rho$ on $\widehat{V}_\rho$, and $q^{\operatorname{Re}\rho} = |q^\rho| = q^{j/2}$ gives $\operatorname{Re}\rho = j/2$. For $u \in \widehat{V}_\rho$, $v \in \widehat{V}_{\rho'}$, $\widehat{h}(\Theta u, v) + \widehat{h}(u, \Theta v) = (\rho + \bar{\rho}')\widehat{h}(u, v)$, which is 0 if $\rho \neq \rho'$ (distinct summands are orthogonal) and $2\operatorname{Re}\rho\, \widehat{h}(u, v) = j\, \widehat{h}(u, v)$ if $\rho = \rho'$. □

Thus a positive Frobenius polarization of weight j suspends to a positive infinitesimal polarization of weight j, the hypothesis of Proposition 3.3, on a spectral core whose spectrum is the full periodic zero divisor of $\det(1 - \mathrm{F}q^{-s})$. The converse holds as well: the infinitesimal identity is the logarithm of $\mathrm{F}^*\mathrm{F} = q^j$.

Proposition 4.17. *Let $q > 1$ and j be real, V a finite-dimensional complex vector space, $\mathrm{F} : V \to V$ invertible, $(\widehat{V}, \Theta)$ its logarithmic suspension, $\widehat{h}$ a positive infinitesimal polarization of weight j on $(\widehat{V}, \Theta)$, and h the restricted form of $\widehat{h}$ for some choice (ρ_α) (Definition 4.15). Then h is a positive Frobenius polarization of weight j on (V, F).*

Proof. By Proposition 3.3, Θ is diagonalizable with eigenvalues of real part $j/2$; so every $N_\alpha = 0$ and $\mathrm{F} = \alpha = q^{\rho_\alpha}$ on $V_{(\alpha)}$, with $|\alpha| = q^{j/2}$. Then $h(\mathrm{F}u, \mathrm{F}v) = \alpha\bar{\beta} h(u, v)$ for $u \in V_{(\alpha)}$, $v \in V_{(\beta)}$, which is $q^j h(u, v)$ when $\alpha = \beta$ and $0 = q^j \cdot 0$ otherwise. □

4.5. **The multiplicative algebra form.** It is natural to expect that the positivity Weil proves for curves is literally an instance of an additive relation $\theta^\dagger = j - \theta$ for the generator θ of a flow, the form in which positivity enters the arithmetic axioms (Section 5); in fact the Rosati involution supplies the multiplicative relation $\pi\pi^\dagger = q$ on the endomorphism algebra of the Jacobian, and the additive relation is obtained from it by functional calculus inside the same algebra, once the multiplicative relation has itself forced purity. This subsection makes the passage exact.

Definition 4.18. Let $q > 1$ and j be real, V a finite-dimensional complex vector space, and $\mathrm{F} : V \to V$ a linear map.

(a) (V, F) satisfies the *multiplicative algebra form AW4-E_{F} of weight j* if there are a subalgebra $A \subset \mathrm{End}_{\mathbb{C}}(V)$ containing id_V and F and a positive trace involution $(\tau, \dagger)$ on A over $k = \mathbb{C}$ (Definition 4.7) with $\mathrm{F}\mathrm{F}^\dagger = q^j$.

(b) If F is invertible, a *sheet* of its logarithmic suspension $(\widehat{V}, \Theta)$ (Definition 4.15) is a map σ from the set of eigenvalues of F to $\mathbb{C}$ with $\sigma(\alpha) \in \Sigma_q(\alpha)$ for every eigenvalue α.

(c) The *sheet space* of a sheet σ is $\widehat{V}_\sigma = \bigoplus_\alpha \widehat{V}_{\sigma(\alpha)} \subset \widehat{V}$, and its *sheet map* $\iota_\sigma : V \to \widehat{V}_\sigma$ sends $v \in V_{(\alpha)}$ to its copy in $\widehat{V}_{\sigma(\alpha)} = V_{(\alpha)}$.

(d) The *interpolation polynomial* of a sheet σ is the polynomial $p_\sigma \in \mathbb{C}[X]$ of degree less than the number of eigenvalues of F with $p_\sigma(\alpha) = \sigma(\alpha)$ for every eigenvalue α.

Distinct eigenvalues α have distinct $\sigma(\alpha)$, because $q^{\sigma(\alpha)} = \alpha$; so the sum in (c) is direct, ι_σ is an isomorphism, and p_σ exists and is unique by Lagrange interpolation. For $m \in \mathbb{Z}$ the map $\sigma + 2\pi i m/\log q$ is again a sheet, and $\widehat{V}$ is the direct sum of the sheet spaces of the sheets $\sigma + 2\pi i m/\log q$, $m \in \mathbb{Z}$: every ρ with $q^\rho = \alpha$ equals $\sigma(\alpha) + 2\pi i m/\log q$ for exactly one m. For a subalgebra $A \subset \mathrm{End}_{\mathbb{C}}(V)$ we write $A^t = \{a^t : a \in A\} \subset \mathrm{End}_{\mathbb{C}}(V^*)$ for the algebra of transposes acting on the dual space V^*.

Proposition 4.19. *Let $q > 1$ and j be real, V a finite-dimensional complex vector space, $\mathrm{F} : V \to V$ linear, and suppose that (V, F) satisfies AW4-E_{F} of weight j (Definition 4.18) with the algebra A and the positive trace involution $(\tau, \dagger)$. Let $(\widehat{V}, \Theta)$ be the logarithmic suspension of (V, F).*

(i) *F is invertible, $\mathrm{F}^{-1} = q^{-j}\mathrm{F}^\dagger$ lies in A, F is diagonalizable, and every eigenvalue α of F satisfies $|\alpha| = q^{j/2}$.*

(ii) *For every sheet σ, the element $\theta_\sigma = p_\sigma(\mathrm{F})$ of A satisfies $\Theta\iota_\sigma = \iota_\sigma\theta_\sigma$ and $\theta_\sigma^\dagger = j - \theta_\sigma$.*

(iii) *Every vector of $\widehat{V}$ lies in a finite-dimensional Θ-stable subspace $\widehat{V}'$ which carries a subalgebra $A' \subset \mathrm{End}_{\mathbb{C}}(\widehat{V}')$ containing $\mathrm{id}_{\widehat{V}'}$ and $\theta' = \Theta|_{\widehat{V}'}$, and a positive trace involution $(\tau', \dagger')$ on A' over $\mathbb{C}$ with $\theta'^{\dagger'} = j - \theta'$.*

(iv) *The dual (V^*, F^t) satisfies AW4-E_{F} of weight j with the algebra A^t, the trace $a^t \mapsto \tau(a)$, and the involution $a^t \mapsto (a^\dagger)^t$.*

Proof of (i). Since $\mathrm{F}\mathrm{F}^\dagger = q^j\,\mathrm{id}_V$ with $q^j \neq 0$ and V is finite-dimensional, F is invertible with inverse $q^{-j}\mathrm{F}^\dagger \in A$. The inclusion $A = A \otimes_{\mathbb{C}} \mathbb{C} \to \mathrm{End}_{\mathbb{C}}(V)$ is an injective unital homomorphism, so Proposition 4.8(ii) with $k = \mathbb{C}$ and $g = \mathrm{F}$ shows that F is diagonalizable with eigenvalues of modulus $q^{j/2}$. □

Proof of (ii). By (i), $V_{(\alpha)} = \ker(\mathrm{F} - \alpha)$, so the nilpotent map N_α of Definition 4.15 vanishes and $\Theta = \sigma(\alpha)$ on $\widehat{V}_{\sigma(\alpha)}$, while $p_\sigma(\mathrm{F})$ acts on $V_{(\alpha)}$ by $p_\sigma(\alpha) = \sigma(\alpha)$; hence $\Theta\iota_\sigma = \iota_\sigma\theta_\sigma$. The element θ_σ lies in A because A contains id_V and F. The involution fixes id_V: from $(ab)^\dagger = b^\dagger a^\dagger$ with $a = \mathrm{id}_V$ and $b = \mathrm{id}_V^\dagger$ one gets $\mathrm{id}_V = \mathrm{id}_V\mathrm{id}_V^\dagger = \mathrm{id}_V^\dagger$, using $b^\dagger = \mathrm{id}_V$. Since $\dagger$ is additive, antilinear, and reverses products, $p(\mathrm{F})^\dagger = \bar{p}(\mathrm{F}^\dagger)$ for every $p \in \mathbb{C}[X]$, where $\bar{p}$ has the conjugate

coefficients. By (i), $\mathrm{F}^\dagger = q^j\mathrm{F}^{-1}$ acts on $V_{(\alpha)}$ by $q^j\alpha^{-1} = \bar\alpha$, because $|\alpha|^2 = q^j$. So $\theta_\sigma^\dagger$ acts on $V_{(\alpha)}$ by $\bar p_\sigma(\bar\alpha) = \overline{\sigma(\alpha)}$. Every $\rho \in \Sigma_q(\alpha)$ has $q^{\mathrm{Re}\,\rho} = |\alpha| = q^{j/2}$, so $\mathrm{Re}\,\sigma(\alpha) = j/2$ and $\overline{\sigma(\alpha)} = j - \sigma(\alpha)$. Hence $\theta_\sigma^\dagger$ and $j - \theta_\sigma$ agree on every $V_{(\alpha)}$, hence on V; both lie in $A \subset \mathrm{End}_{\mathbb{C}}(V)$, so they are equal. □

Proof of (iii). Fix a sheet σ_0 and put $\sigma_m = \sigma_0 + 2\pi i m/\log q$ for $m \in \mathbb{Z}$. A vector of $\widehat V$ has finitely many nonzero components, so it lies in $\widehat V' = \bigoplus_{|m|\le n} \widehat V_{\sigma_m}$ for some n; this space is finite-dimensional and Θ-stable. Let A' be the set of maps acting on each $\widehat V_{\sigma_m}$, $|m| \le n$, as $\iota_{\sigma_m} a_m \iota_{\sigma_m}^{-1}$ for some $a_m \in A$, a subalgebra of $\mathrm{End}_{\mathbb{C}}(\widehat V')$ isomorphic to the product of $2n+1$ copies of A; put $\tau' = \sum_m \tau(a_m)$ and let $\dagger'$ act componentwise. Every identity in Definition 4.7 holds componentwise, and $\tau'(xx^{\dagger'}) = \sum_m \tau(a_m a_m^\dagger) > 0$ unless every a_m vanishes, so $(\tau', \dagger')$ is a positive trace involution. By (ii), θ' is the element of A' with components θ_{σ_m}, and $\theta'^{\dagger'}$ has components $j - \theta_{\sigma_m}$, that is, $\theta'^{\dagger'} = j - \theta'$. □

Proof of (iv). Transposition $a \mapsto a^t$ is a bijective linear map $A \to A^t$ reversing products, and $(\mathrm{F}^t)^{-1} = (\mathrm{F}^{-1})^t$. Transporting τ and $\dagger$ along it preserves every identity in Definition 4.7: for instance $(a^t b^t)^\dagger = ((ba)^\dagger)^t = (a^\dagger b^\dagger)^t = (b^\dagger)^t (a^\dagger)^t$, and $\tau(a^t(a^\dagger)^t) = \tau(a^\dagger a) = \tau(aa^\dagger) > 0$ for $a \neq 0$. Finally $\mathrm{F}^\dagger\mathrm{F} = q^j$ because $\mathrm{F}^\dagger = q^j\mathrm{F}^{-1}$, so $\mathrm{F}^t(\mathrm{F}^t)^\dagger = (\mathrm{F}^\dagger\mathrm{F})^t = q^j$. □

Item (iii) is the condition AW4-E of Definition 5.5 in Section 5, read on $(\widehat V, \Theta)$ with weight j. The multiplicative form is canonical, while the additive form depends on the choice of sheets, exactly as the flow depends on the logarithm of Frobenius (Proposition 7.22 below); the passage between them uses the purity supplied by the multiplicative form itself (item (i)).

Proposition 4.20. *Let C, $g \ge 1$, J, π, ℓ, and the Rosati involution $a \mapsto a^\dagger$ on $\mathrm{End}^0(J)$ be as in Theorem 4.10, let $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ be a field isomorphism, put $V = V_\ell J \otimes_{\mathbb{Q}_\ell,\iota} \mathbb{C}$, let $\mathrm{F} : V \to V$ be the map induced by π, and let $A \subset \mathrm{End}_{\mathbb{C}}(V)$ be the image of the homomorphism $\mathrm{End}^0(J) \otimes_{\mathbb{Q}} \mathbb{C} \to \mathrm{End}_{\mathbb{C}}(V)$ induced by the action on $V_\ell J$.*

- *(i) This homomorphism is injective.*
- *(ii) There is a unique antilinear map $\dagger$ on A sending the image of $a \otimes z$ to the image of $a^\dagger \otimes \bar z$ ($a \in \mathrm{End}^0(J)$, $z \in \mathbb{C}$), and with τ the trace of endomorphisms of V, the pair $(\tau, \dagger)$ is a positive trace involution on A over $\mathbb{C}$.*
- *(iii) (V, F) satisfies AW4-E_{F} of weight one with this algebra and involution.*

Proof of (i). By Theorem 4.10(i), the map $\mathrm{End}^0(J) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \to \mathrm{End}_{\mathbb{Q}_\ell}(V_\ell J)$ is injective. Tensoring over $\mathbb{Q}_\ell$ with $\mathbb{C}$ (through ι) preserves injectivity, and $\mathrm{End}_{\mathbb{Q}_\ell}(V_\ell J) \otimes_{\mathbb{Q}_\ell} \mathbb{C} = \mathrm{End}_{\mathbb{C}}(V)$. □

Proof of (ii). The map $(a, z) \mapsto a^\dagger \otimes \bar z$ is $\mathbb{Q}$-bilinear, so it induces an additive, antilinear map on $\mathrm{End}^0(J) \otimes_{\mathbb{Q}} \mathbb{C}$, which by (i) defines $\dagger$ on A; it reverses products and is an involution because the Rosati involution does, as one checks on pure tensors. The trace τ satisfies $\tau(xy) = \tau(yx)$. For $a \in \mathrm{End}^0(J)$ let $\mathrm{Tr}(a)$ be its trace on $V_\ell J$; it is rational by Theorem 4.10(i), and $\tau(a \otimes z) = z\,\mathrm{Tr}(a)$. The Rosati involution is the adjoint for the nondegenerate alternating Weil pairing e^λ on $V_\ell J$, $e^\lambda(aa_1, a_2) = e^\lambda(a_1, a^\dagger a_2)$ [Mil08, §14, p. 61], so $a^\dagger$ has the characteristic polynomial of a and $\mathrm{Tr}(a^\dagger) = \mathrm{Tr}(a)$; hence $\tau(x^\dagger) = \overline{\tau(x)}$ on pure tensors and, by additivity, on A. For positivity, let $(a_i)_{1\le i\le n}$ be a $\mathbb{Q}$-basis of $\mathrm{End}^0(J)$ and $G_{ik} = \mathrm{Tr}(a_i a_k^\dagger) \in \mathbb{Q}$. The matrix G is symmetric, since $\mathrm{Tr}(a_k a_i^\dagger) = \mathrm{Tr}((a_i a_k^\dagger)^\dagger)$, and positive definite on $\mathbb{Q}^n$ by Theorem 4.10(ii); as in the proof of Proposition 4.8(i), it is then positive definite on $\mathbb{R}^n$. Every $x \in A$ is the image of $\sum_i a_i \otimes z_i$ for a unique $z \in \mathbb{C}^n$, and $\tau(xx^\dagger) = \sum_{i,k} z_i \bar z_k G_{ik} = u^T G u + v^T G v$ for $z = u + iv$ with $u, v \in \mathbb{R}^n$, which is positive for $z \neq 0$. □

Proof of (iii). A contains the identity and F, the image of $\pi \otimes 1$, and $\mathrm{FF}^\dagger$ is the image of $\pi\pi^\dagger \otimes 1 = q \otimes 1$ by Theorem 4.10(iii). □

Thus Weil's positivity is an instance of the multiplicative algebra form, and by Proposition 4.19 it gives, inside the same algebra, the semisimplicity and purity of π on $V_\ell J \otimes_\iota \mathbb{C}$ (the complex form of Proposition 4.11) and the additive algebra form on the logarithmic suspension of $(V_\ell J \otimes_\iota \mathbb{C}, \pi)$. The passage to H^1 is the one import that remains. It consists of three facts, none re-derived here: $H_1(C_{\mathrm{et}}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell \otimes T_\ell J$ [Mil15, Example 3.8, p. 40]; the canonical isomorphism $H^1(J_{\overline{\mathbb{F}}_q}, \mathbb{Z}_\ell) \cong \mathrm{Hom}(T_\ell J, \mathbb{Z}_\ell)$ and its compatibility with the Galois action [Mil08, Chapter I, Theorem 12.1(a), p. 55, and Remark 12.5, p. 56]; and the isomorphism $H^1(J_{\overline{\mathbb{F}}_q}, \mathbb{Z}_\ell) \to H^1(C_{\overline{\mathbb{F}}_q}, \mathbb{Z}_\ell)$ induced by an Abel–Jacobi map [Mil08, Chapter III, Corollary 9.6, p. 114], which does not depend on the base point, because translations act trivially on $H^1(J)$ (the addition map pulls $x \in H^1$ back to $x \otimes 1 + 1 \otimes x$ [Mil08, Chapter I, (3), p. 55]), and hence commutes with the pullbacks by the q-power Frobenius endomorphisms of C and J, the latter being π. The naturality of these isomorphisms, which turns pullback by π into the transpose of the action of π on $T_\ell J$, is also taken from the source. So $H^1(C_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ is the dual of $V_\ell J$ with Frobenius the transpose of π; in genus 0 both are zero. Granting this, Proposition 4.19(iv) carries the multiplicative form from $V_\ell J$ to H^1 (proof of Theorem 7.24(vi)).

5. Analytic and positivity conditions on Deninger structures

We formulate axioms, following Deninger's conjectural determinant formula [Den10, Conjecture 2, p. 3], his further expectations of a trace, a cup-product pairing, and a positive star [Den10, pp. 4–5], and the reformulation by Flach and Morin [FM20, Definition 2.1, Conjecture 2.3], in a form sufficient for an exact implication to the Riemann hypothesis. This section describes the stronger cup-product, duality, and star package over the uniform spectral formalism of Section 1.2. Its global arithmetic construction with compatible positivity is a target; the suspended finite-field structures are constructed in Theorem 7.24. The values of a Deninger cohomology theory (Definition 1.21) are Deninger structures (Definition 1.11); the conditions below are imposed on a single such structure, polarized or not, and on its Weil datum.

For a number field K we write $\overline{X}_K = (\operatorname{Spec} \mathcal{O}_K, X_{K,\infty})$ for the Arakelov compactification of $\operatorname{Spec} \mathcal{O}_K$: an arithmetic pair, not a scheme. Its dimension is the Krull dimension of $\mathcal{O}_K$, namely one.

The *arithmetic test category* is the full subcategory $\overline{\mathrm{Sch}}_1$ of $\overline{\mathrm{Sch}}^{\mathrm{proj}}$ (Definition 1.19(e)), whose objects are the finite disjoint unions of the $\overline{X}_K$.

Compare the Arakelov compactifications of [FM20, Notation 2.2] and [Den23, §2].

Proposition 5.1 (Number fields and arithmetic curves)**.** *Let $\mathcal{N}$ be the category of number fields with field homomorphisms as morphisms. The assignment $K \mapsto \overline{X}_K$, $(\alpha : K \to L) \mapsto \operatorname{Spec}(\alpha|_{\mathcal{O}_K})$, is an equivalence from $\mathcal{N}^{\mathrm{op}}$ onto the full subcategory of connected objects of $\overline{\mathrm{Sch}}_1$.*

Proof. A ring homomorphism $\mathcal{O}_K \to \mathcal{O}_L$ extends uniquely to $K \to L$, and a field homomorphism $K \to L$ carries integral elements to integral elements, hence $\mathcal{O}_K$ into $\mathcal{O}_L$. Thus $\mathrm{Hom}_{\mathcal{N}}(K, L) = \mathrm{Hom}(\mathcal{O}_K, \mathcal{O}_L) = \mathrm{Hom}(\operatorname{Spec} \mathcal{O}_L, \operatorname{Spec} \mathcal{O}_K)$. The map on infinite places is induced by the scheme morphism, so it adds no choice. Every connected object of $\overline{\mathrm{Sch}}_1$ is some $\overline{X}_K$. □

In particular the morphisms $\overline{X}_L \to \overline{X}_K$ between connected objects are exactly those induced by field embeddings $K \hookrightarrow L$, with the corresponding finite map $\operatorname{Spec} \mathcal{O}_L \to \operatorname{Spec} \mathcal{O}_K$ and map of infinite places, and the automorphisms of $\overline{X}_K$ form $\mathrm{Aut}(K)$.

A contravariant theory on $\overline{\mathrm{Sch}}_1$ already contains $\overline{X}_{\mathbb{Q}}$, and all number fields are needed for the Galois-functoriality test of Remark 9.21. For the implication to RH alone, the single object

$\overline{X}_{\mathbb{Q}}$ suffices. Larger categories appear in [Den10, Conjecture 2, p. 3] (separated $\mathbb{Z}$-schemes of finite type) and in [FM20, Conjecture 2.3, p. 522] (arithmetic schemes, with cohomology with compact supports, together with Arakelov compactifications of projective flat regular schemes); they extend the minimal number-field package and motivate the uniform source category of Definition 1.19.

It is natural to expect such a larger theory to carry a Künneth formula, as Weil cohomologies over a field do; in fact its natural strong forms are inconsistent with the axioms below. No Künneth formula is imposed in the axioms, and the test category $\overline{\mathrm{Sch}}_1$ is not presented as a scheme category over a "field with one element". Section 8 states the Künneth conjecture in the form that survives, records its refutations (no formula over $\operatorname{Spec}\mathbb{Z}$, even in derived form; no spectral core, no compact resolvent, and no determinant on the tensor square of an arithmetic curve), and proves the Hodge index statement on the arithmetic surface that corresponds to the positivity in Weil's proof.

Spectral cores are defined in the introduction (Definition 1.1); by Proposition 4.16(ii), a logarithmic suspension is one.

Definition 5.2. Let (V,Θ) be a spectral core with multiplicities m.

(a) A *spectral cut* on a nonempty open set $U\subset\mathbb{C}$ is a choice, for each $\rho\in\operatorname{spec}\Theta$, of a holomorphic branch L_ρ of $\log((s-\rho)/2\pi)$ on U (so $U\cap\operatorname{spec}\Theta=\emptyset$). For $s\in U$ and $w\in\mathbb{C}$ put $((s-\rho)/2\pi)^{-w}=e^{-wL_\rho(s)}$.

(b) The *spectral zeta function* for this cut is

$$Z(w;s)=\sum_{\rho\in\operatorname{spec}\Theta} m(\rho)\Big(\frac{s-\rho}{2\pi}\Big)^{-w}\qquad (s\in U),$$

wherever the series converges absolutely.

(c) (V,Θ) is *zeta-regularizable* for the cut if, for every $s\in U$, the series converges for $\operatorname{Re}w$ large; there is $\varepsilon=\varepsilon(s)>0$ such that $Z(w;s)$ continues to a meromorphic function of w on the half-plane $\operatorname{Re}w>-\varepsilon$ that is regular at $w=0$; the function $s\mapsto\partial_w Z(0;s)$ is holomorphic on U; and $s\mapsto\exp(-\partial_w Z(0;s))$ continues to an entire function of s.

(d) The *regularized determinant* for the cut is that entire function, $\det_\infty\big((s-\Theta)/2\pi\big)=\exp(-\partial_w Z(0;s))$.

Two half-planes $\operatorname{Re}w>-\varepsilon$ and $\operatorname{Re}w>-\varepsilon'$ are nested, and each contains $w=0$ and the half-plane of absolute convergence. By the identity theorem the continuation in (c) is therefore unique, so $\partial_w Z(0;s)$ and $\det_\infty$ depend only on the spectral cut.

Whenever a regularized determinant is used below, a spectral cut is fixed and recorded as part of the data. If V is finite-dimensional, then for every spectral cut $\partial_w Z(0;s)=-\sum_\rho m(\rho)L_\rho(s)$, so $\det_\infty((s-\Theta)/2\pi)=\prod_\rho((s-\rho)/2\pi)^{m(\rho)}$ independently of the cut. In particular, if $V=\mathbb{C}$ and $\Theta=\rho_0$, then $\det_\infty((s-\Theta)/2\pi)=(s-\rho_0)/2\pi$. For infinite spectra the determinant depends on the cut; the one cut used below is Deninger's [Den94, (1.1), p. 3].

Definition 5.3. Let (V,Θ) be a spectral core whose spectrum lies in a vertical strip $\{a\le\operatorname{Re}s\le b\}$, and let Log be the principal branch of the logarithm on $\mathbb{C}\setminus(-\infty,0]$.

(a) The *principal cut* of (V,Θ) is the spectral cut on $U_{\mathrm{pr}}=\mathbb{C}\setminus\bigcup_{\rho\in\operatorname{spec}\Theta}\big(\rho+(-\infty,0]\big)$ given by $L_\rho(s)=\operatorname{Log}((s-\rho)/2\pi)$.

(b) The *reflected cut* of (V,Θ) is the spectral cut on $U_{\mathrm{rf}}=\mathbb{C}\setminus\bigcup_{\rho\in\operatorname{spec}\Theta}\big(\rho+[0,\infty)\big)$ given by the branch $L_\rho(s)$ of $\log((s-\rho)/2\pi)$ with imaginary part in $[0,2\pi)$.

The sets U_{pr} and U_{rf} are open and contain the half-planes $\operatorname{Re}s>b$ and $\operatorname{Re}s<a$ respectively: only finitely many eigenvalues have imaginary part in a given bounded interval, so each union

of rays is locally finite, hence closed. The principal cut is Deninger's convention $-\pi < \arg \le \pi$, restricted to U_{pr}; there it agrees with his pointwise definition. The reflected cut computes Deninger's $\det_\infty(-\Theta)$ [Den94, (1.3)–(1.7), pp. 4–6]; the regularized determinant with the principal cut is also [FM20, Definition 2.1, p. 521].

The internal unital axioms apply in weight zero. For compactified number rings $\overline{X}_K$, $d = 1$; the finite-field calibration uses the axioms with $d = \dim X$, $w = 0$, and a periodic unit object; arithmetic schemes use them with $d \ge 2$. Pure coefficients of weight w use graded spectral realizations of type $(1, w)$, as in Remark 7.31; their products have separate targets.

Definition 1.3 is stated in the introduction.

The q-periodic unit is Deninger's space R_p of finite Fourier series on $\mathbb{R}/(\log p)\mathbb{Z}$ with $\Theta = d/dy$, complexified, for $q = p$ [Den98, pp. 167–168 and Proposition 3.1].

Definition 1.4 and the core axioms AW1–AW4 of Definition 1.5 are stated in the introduction.

Definition 5.4 (Additional analytic conditions)**.** Let a Weil datum of type $(d, 0)$ be as in Definition 1.4. It satisfies AW5 or AW6 when the corresponding condition below holds.

AW5 *(Hilbert realization.)* Each h_j is positive definite; let $\mathscr{H}^j$ be the completion of (V^j, h_j). The normalized flow $U_j(t) = e^{-jt/2}\phi^t$ extends to a strongly continuous unitary group on $\mathscr{H}^j$ whose generator A_j is skew-adjoint and has V^j as a core; put $T_j = -iA_j$, so that T_j is self-adjoint and the closure of Θ_j is $j/2 + iT_j$.

AW6 *(No loss on completion; schematic.)* T_j has pure point spectrum equal to $\{(\rho - j/2)/i : \rho \in \operatorname{spec} \Theta_j\}$, with the same finite multiplicities and no finite accumulation; its counting function satisfies $N_j(R) = O(R^M)$ for some M; and for test functions f in a class $\mathcal{S}_X$ the trace $\operatorname{tr} f(T_j)$ equals the explicit-formula distribution of the object, with an archimedean regularization.

The endpoint clause of AW1 is Deninger's normalization of H^0 and H^2 [Den98, (3), p. 168]; the duality of AW3 and the star law of AW4 are [Den98, pp. 168–169 and p. 172], [Den10, pp. 4–5], and [FM20, Conjecture 2.3(iii)–(iv), p. 522]; the Hilbert completion of AW5 is [Den94, p. 33]. Positive-weight coefficients use separate spectral realizations, not these internal AW axioms. Condition AW6 is schematic as stated. For $\overline{X}_K$ its trace-formula clause is made precise by the axiom AW6′ of Definition 5.9 below, and its first clause follows from AW5 (Proposition 7.46); Theorems 6.3 and 7.4 do not use AW5 or AW6. The divisor identity AW2 is imposed. Over $\mathbb{F}_q$ rationality is derived from the Lefschetz trace formula; over $\overline{X}_K$ the trace-formula axiom AW6′ is, given AW1, equivalent to AW2 (Theorem 7.43), so replacing AW2 by AW6′ changes the form of the assumption, not its strength. The two further forms of the positivity axiom, and the determinant form of AW2, are separate definitions.

Definition 5.5. Let $(V^\bullet, \Theta, \cup, \mathrm{Tr}, *)$ be a Weil datum of type $(d, 0)$ and $0 \le j \le 2d$.

(a) The datum satisfies *AW4-V in degree j* if (V^j, Θ_j) admits a positive infinitesimal polarization of weight j (Definition 1.7).

(b) The datum satisfies *AW4-E in degree j* if every vector of V^j lies in a finite-dimensional Θ_j-stable subspace V' which carries a subalgebra $A_{V'} \subset \mathrm{End}_{\mathbb{C}}(V')$ containing $\mathrm{id}_{V'}$ and $\theta_{V'} = \Theta_j|_{V'}$, and a positive trace involution $(\tau_{V'}, \dagger)$ on $A_{V'}$ over $k = \mathbb{C}$ (Definition 4.7) with $\theta_{V'}^\dagger = j - \theta_{V'}$.

Definition 5.6. Let $(V^j, \Theta_j)_{0 \le j \le 2d}$ be graded spectral cores, and let $Z \not\equiv 0$ be meromorphic on $\mathbb{C}$. They satisfy *AW-det for Z* if, for each j, a spectral cut for (V^j, Θ_j) is fixed as part of the data, (V^j, Θ_j) is zeta-regularizable for it (Definition 5.2), the entire function $D_j(s) = \det_\infty((s - \Theta_j)/2\pi)$ satisfies $\operatorname{div} D_j = \mathrm{Sp}_j$, and there are recorded constants $a, b \in \mathbb{C}$, the *normalization*

term e^{a+bs}, with

$$e^{a+bs}\,Z(s) = \prod_{j=0}^{2d} D_j(s)^{(-1)^{j+1}}. \tag{AW-det}$$

Taking divisors in (AW-det) gives the spectral divisor identity of Definition 1.8. For an unweighted Weil datum this is AW2. The converse fails: the divisor identity asks nothing of regularization. All zero-location theorems below use AW2, or AW6′, which is equivalent to it given AW1 (Theorem 7.43), and never AW-det, so no value of the normalization term enters them; AW-det records the stronger, determinant-level expectation of Deninger [Den10, Conjecture 2, p. 3] and Flach–Morin [FM20, Conjecture 2.3]. The normalization term for $\overline{X}_K$ is computed in Theorem 7.37.

For a number field K the endpoint normalization is $V^0 = \mathbb{C}\cdot 1$ with $\Theta_0 = 0$ and $V^2 \cong \mathbb{C}(-1)$.

A Weil datum on $\overline{X}_K$ satisfying AW1–AW4 for $(\Lambda_K, \mathbb{1})$ is a polarized Deninger structure in $\mathrm{Den}_1^{\mathrm{pol}}(\mathbb{1})$, with a chosen polarization, satisfying AW2 for Λ_K. A Deninger cohomology theory (Definition 1.21) restricts to a functor on $\overline{\mathrm{Sch}}_1$ whose value on each $\overline{X}_K$ is such a structure without the polarization. The trace-formula clause of AW6 is made precise for $\overline{X}_K$ by the axiom AW6′ below. It tests the flow against smooth compactly supported functions of the time t and compares the result with the geometric side of the explicit formula.

Definition 5.7. Let (V, Θ) be a spectral core with multiplicity function m.

(a) The *transform* of $\varphi \in C_c^\infty(\mathbb{R})$ is the entire function $\Phi(s) = \int_{\mathbb{R}} \varphi(t) e^{st}\,dt$.

(b) The *two-sided class* is $\mathcal{S} = C_c^\infty(\mathbb{R})$, the *one-sided class* is $\mathcal{S}^+ = C_c^\infty((0,\infty))$, for $T_0 \in \mathbb{R}$ the *tail class* is $\mathcal{S}_{>T_0} = C_c^\infty((T_0,\infty))$, and for $L > 0$ the *window classes* are $\mathcal{S}_{(0,L)} = C_c^\infty((0,L))$ and $\mathcal{S}_{(-L,L)} = C_c^\infty((-L,L))$.

(c) (V, Θ) satisfies the *growth condition* (G) if $\operatorname{spec}\Theta$ lies in a vertical strip $\{a \le \operatorname{Re} s \le b\}$ and $\sum_{|\rho|\le R} m(\rho) = O(R^M)$ as $R \to \infty$ for some M.

(d) For $\varphi \in \mathcal{S}$ such that the series converges absolutely, the *distributional trace* of the flow of (V, Θ) at φ is $\mathrm{Tr}_{\mathrm{dis}}(V)(\varphi) = \sum_\rho m(\rho)\Phi(\rho)$.

Under $x = e^t$, the function $x \mapsto \varphi(\log x)$ lies in $C_c^\infty(\mathbb{R}_{>0})$ and Φ is its Mellin transform $\int_0^\infty \varphi(\log x) x^{s-1}\,dx$; the distributional trace is Deninger's [Den98, (4) and the display after it, p. 170]. The source writes the sum over ρ without an explicit weight; the weight $m(\rho)$ is the standard reading, and it is the one that the proof of Theorem 7.41 establishes.

Definition 5.8. Let K be a number field with r_1 real and r_2 complex places and discriminant D_K, and let $\varphi \in \mathcal{S}$.

(a) The *archimedean terms* at φ are

$$W_{\mathbb{R}}(\varphi) = \log\pi\cdot\varphi(0) + \int_0^\infty \Big[\frac{\varphi(t)+e^{-t}\varphi(-t)}{1-e^{-2t}} - \frac{e^{-2t}\varphi(0)}{t}\Big]dt,$$

$$W_{\mathbb{C}}(\varphi) = 2\log(2\pi)\cdot\varphi(0) + \int_0^\infty \Big[\frac{\varphi(t)+e^{-t}\varphi(-t)}{1-e^{-t}} - \frac{2e^{-t}\varphi(0)}{t}\Big]dt.$$

(b) The *geometric side* of the explicit formula of K at φ is

$$\begin{aligned} G_K(\varphi) = {} & -\log|D_K|\,\varphi(0) + r_1 W_{\mathbb{R}}(\varphi) + r_2 W_{\mathbb{C}}(\varphi) \\ & + \sum_{\mathfrak{p}} \log N\mathfrak{p} \sum_{k\ge 1} \big(\varphi(k\log N\mathfrak{p}) + N\mathfrak{p}^{-k}\varphi(-k\log N\mathfrak{p})\big), \end{aligned}$$

the sum running over the nonzero prime ideals of $\mathcal{O}_K$.

The prime sum is finite because φ has compact support; both integrands are bounded near $t = 0$, where the two terms of size $\varphi(0)/t$ cancel, and are integrable at infinity. The geometric side involves only the prime ideals, the archimedean places, and the discriminant of K, not its zeros. For $\varphi \in \mathcal{S}^+$ it reduces to

$$G_K(\varphi) = \sum_{\mathfrak{p}} \log N\mathfrak{p} \sum_{k\geq 1} \varphi(k \log N\mathfrak{p}) + r_1 \int_0^\infty \frac{\varphi(t)\,dt}{1-e^{-2t}} + r_2 \int_0^\infty \frac{\varphi(t)\,dt}{1-e^{-t}}. \tag{5.1}$$

Definition 5.9. Let K be a number field, let $(V^\bullet, \Theta, \cup, \mathrm{Tr}, *)$ be a Weil datum of type $(1,0)$ on $\overline{X}_K$, and let $\mathcal{S}' \subset \mathcal{S}$.

(a) The datum satisfies *AW6′ for* $(\Lambda_K, \nVdash)$ *on* $\mathcal{S}'$ if each (V^j, Θ_j) satisfies (G) and, for every $\varphi \in \mathcal{S}'$, the three distributional traces converge absolutely and

$$\sum_{j=0}^{2} (-1)^j \,\mathrm{Tr}_{\mathrm{dis}}(V^j)(\varphi) = G_K(\varphi). \tag{AW6′}$$

AW6′ without qualification means $\mathcal{S}' = \mathcal{S}$.

(b) Suppose that each star form h_j is positive definite and that the datum satisfies AW5, with unitary groups U_j on the completions $\mathscr{H}^j$. The datum satisfies *AW6′ in Hilbert form on* $\mathcal{S}'$ if each (V^j, Θ_j) satisfies (G) and, for every $\varphi \in \mathcal{S}'$, the operators $A_\varphi^{(j)} = \int_{\mathbb{R}} \varphi(t) e^{jt/2} U_j(t)\,dt$ (strong integrals) are of trace class and $\sum_{j=0}^{2} (-1)^j \,\mathrm{tr}\, A_\varphi^{(j)} = G_K(\varphi)$.

In the functional calculus of the self-adjoint operator T_j of AW5, $A_\varphi^{(j)} = f_j(T_j)$ with $f_j(\tau) = \Phi(\frac{j}{2} + i\tau)$. So a trace "$\mathrm{tr}\, f(T_j)$" of the schematic AW6 must use a test function depending on the degree j, and the terms $\Phi(0)$ and $\Phi(1)$ of the explicit formula are the traces in degrees 0 and 2. The test function must also be the transform of a compactly supported φ, not an arbitrary Schwartz function of τ (Proposition 7.53). The form (a) uses only the spectral cores, with no Hilbert space, positivity, or star; the form (b) presupposes AW5, which presupposes that h_j is positive definite and then yields AW4-V (Proposition 7.46).

Remark 5.10. The following comments record the scope of the axioms. The quaternionic refinement $*_1^2 = -1$ in degree one [Den23, 2.7, p. 3] is compatible with AW4 and is used only in Proposition 7.16. By AW1 and AW3, for eigenvectors $u \in V^j$, $v \in V^{2d-j}$ with eigenvalues ρ, σ one has $d\,\mathrm{Tr}(u \cup v) = \mathrm{Tr}(\Theta(u \cup v)) = (\rho + \sigma)\,\mathrm{Tr}(u \cup v)$, so the pairing is supported on $\sigma = d - \rho$; perfectness then gives $m_j(\rho) = m_{2d-j}(d - \rho)$, the spectral shadow of the functional equation. Spectral duality determines the reflected divisor, not the nonvanishing factors of a functional equation; conductor and epsilon data are extra normalization data. Axioms AW5–AW6 are not used in Theorems 6.3 and 7.4; AW5 presupposes that h_j is positive definite, and with its unitary flow it yields AW4-V (Proposition 7.46); both make the operator and trace-formula comparisons of Sections 12 and 13 meaningful. Finally, the data must be constructed from arithmetic input: defining a vector for each known zero is a formal encoding, not the desired construction (Propositions 6.6 and 7.6).

It is natural to expect that, as over a finite field, the zeta function of a higher-dimensional arithmetic object has all its zeros on one line, and that zeros and poles coming from different degrees may cancel in the alternating product; in fact under AW4 each degree has its own line, the lines are disjoint, and no cancellation occurs (Theorem 7.28, Propositions 7.27 and 7.30).

Remark 5.11. Over $\mathbb{F}_q$ the zeta function is a rational function of q^{-s} and purity is read off from characteristic polynomials of Frobenius. Over a number field there is no single q, and Λ_K is not such a rational function: AW2 replaces the cohomological expression of $Z(X,t)$ by the divisor

of a flow, AW3 replaces Poincaré duality for Frobenius by duality for a flow, and AW4 replaces the positive Frobenius polarization of Definition 4.1 by an infinitesimal one, exactly as in the suspension of Proposition 4.16. Table 1 in Section 1 lists the correspondence. For a pure motive of weight w, the expected critical line of its L-function is $\operatorname{Re} s = (w+1)/2$; this is the separate graded spectral convention.

6. Purity and the conditional Riemann hypothesis

The purity statement, Proposition 3.3, is stated in Section 3, before its first use.

Proposition 6.1 ([Den10, p. 5])**.** *Let $d \geq 0$ be an integer, and let $(V^\bullet, \Theta, \cup, \operatorname{Tr}, *)$ be a Weil datum of type $(d,0)$ (Definition 1.4) satisfying the derivation rule of AW1, the trace equivariance of AW3, and the star law of AW4. Then for each j the star form $h_j(u,v) = \operatorname{Tr}(u \cup *_j v)$ satisfies $h_j(\Theta_j u, v) + h_j(u, \Theta_j v) = j\, h_j(u,v)$ for all $u, v \in V^j$.*

Proof. The star law gives $*_j(\Theta_j v) = \Theta_{2d-j}(*_j v) - (d-j) *_j v$, and $\operatorname{Tr}(u \cup *_j \Theta_j v) = h_j(u, \Theta_j v)$ by definition. Hence, using the derivation rule and then trace equivariance,

$$\begin{aligned} h_j(\Theta_j u, v) + h_j(u, \Theta_j v) &= \operatorname{Tr}(\Theta_j u \cup *_j v) + \operatorname{Tr}(u \cup \Theta_{2d-j} *_j v) - (d-j)\operatorname{Tr}(u \cup *_j v) \\ &= \operatorname{Tr}\big(\Theta_{2d}(u \cup *_j v)\big) - (d-j) h_j(u,v) = d h_j(u,v) - (d-j) h_j(u,v). \qquad \square \end{aligned}$$

The computation is Deninger's [Den10, p. 5], [Den94, (7.12)–(7.13), p. 33], [Den23, 2.7, p. 3]. Thus under AW4 the star form h_j is a positive infinitesimal polarization of weight j on (V^j, Θ_j): the star form of the positivity axiom implies AW4-V in every degree. The algebra form does too.

Proposition 6.2. *Let $(V^\bullet, \Theta, \cup, \operatorname{Tr}, *)$ be a Weil datum of type $(d,0)$, and let $0 \leq j \leq 2d$.*

- *(i) If the datum satisfies AW4-E in degree j (Definition 5.5), then Θ_j is diagonalizable on every generalized eigenspace, every eigenvalue ρ of Θ_j satisfies $\operatorname{Re}\rho = j/2$, and the datum satisfies AW4-V in degree j.*
- *(ii) If the datum satisfies AW4-V in degree j, then it satisfies AW4-E in degree j.*
- *(iii) If the datum satisfies the derivation rule of AW1, satisfies AW3, and satisfies AW4-V in every degree, then there are antilinear maps $*'_j : V^j \to V^{2d-j}$ such that the datum with stars $*'_j$ satisfies AW4, with star forms equal to the given polarizations.*

Proof of (i). Let V', $A = A_{V'}$, $\theta = \theta_{V'}$, and $(\tau, \dagger)$ be as in Definition 5.5. Put $\langle a, b\rangle = \tau(ab^\dagger)$ on A. It is linear in a and antilinear in b, Hermitian because $\overline{\langle b, a\rangle} = \tau((ba^\dagger)^\dagger) = \tau(ab^\dagger)$, and positive definite by hypothesis. Let $L_\theta(a) = \theta a$. Then $\langle \theta a, b\rangle = \tau(\theta a b^\dagger) = \tau(ab^\dagger\theta) = \tau(a(\theta^\dagger b)^\dagger) = \langle a, \theta^\dagger b\rangle$, so the adjoint of L_θ is $L_{\theta^\dagger} = j - L_\theta$, and $\langle\,,\rangle$ is a positive infinitesimal polarization of weight j on (A, L_θ). By Proposition 3.3(iii) and (i) (with $V = A$, which is finite-dimensional), L_θ is diagonalizable with eigenvalues of real part $j/2$, so its minimal polynomial m is separable with roots on $\operatorname{Re} s = j/2$. Since A is unital, $p(L_\theta) = 0$ if and only if $p(\theta) = 0$ (evaluate at 1, and conversely $p(L_\theta)a = p(\theta)a$), so m is the minimal polynomial of θ in A, hence in $\operatorname{End}_{\mathbb{C}}(V')$. Therefore θ is diagonalizable on V' with eigenvalues of real part $j/2$. Let $v \in V^j_{\Theta \sim \rho}$ be nonzero, with $(\Theta_j - \rho)^k v = 0$, and choose $V' \ni v$ as above. Then ρ is an eigenvalue of θ, so $\operatorname{Re}\rho = j/2$, and since θ is diagonalizable on V', $(\theta - \rho)^k v = 0$ forces $(\theta - \rho)v = 0$. So $V^j = \bigoplus_\rho \ker(\Theta_j - \rho)$ with $\operatorname{Re}\rho = j/2$. Let h be the orthogonal sum of arbitrary positive definite Hermitian forms on the finite-dimensional eigenspaces. For eigenvectors u, v with eigenvalues ρ, ρ', $h(\Theta u, v) + h(u, \Theta v) = (\rho + \bar\rho')h(u,v)$, which is 0 if $\rho \neq \rho'$ and $jh(u,v)$ if $\rho = \rho'$; so h is a positive infinitesimal polarization of weight j. $\square$

Proof of (ii). Let h be a positive infinitesimal polarization of weight j on (V^j, Θ_j). Every vector of V^j lies in a finite sum V' of generalized eigenspaces, which is finite-dimensional and Θ_j-stable. Take $A_{V'} = \mathrm{End}_{\mathbb{C}}(V')$, $\tau_{V'}$ the trace, and $a^\dagger$ the $h|_{V'}$-adjoint of a. Then $(\tau_{V'}, \dagger)$ is a positive trace involution: in an h-orthonormal basis $a^\dagger$ is the conjugate transpose and $\tau_{V'}(aa^\dagger)$ is the sum of the squared moduli of the entries of a. Identity (1.1) says $\theta^\dagger_{V'} = j - \theta_{V'}$. □

Proof of (iii). Fix j and a positive infinitesimal polarization h of weight j on V^j. By Proposition 3.3, V^j is the direct sum of the eigenspaces $\ker(\Theta_j - \rho)$, which are h-orthogonal. For v in the ρ-eigenspace the linear form $u \mapsto h(u, v)$ vanishes on the other eigenspaces, and by AW3 the pairing B_j restricts to a perfect pairing of $V^j_{\Theta\sim\rho}$ with $V^{2d-j}_{\Theta\sim d-\rho}$ and vanishes between $V^j_{\Theta\sim\rho'}$ and $V^{2d-j}_{\Theta\sim d-\rho}$ for $\rho' \neq \rho$ (Remark 5.10). So there is a unique $*'_j v \in V^{2d-j}_{\Theta\sim d-\rho}$ with $B_j(u, *'_j v) = h(u, v)$ for all $u \in V^j$; the map $*'_j$ is antilinear, and its star form is h. The derivation rule and the trace equivariance give $B_j(\Theta u, x) + B_j(u, \Theta x) = dB_j(u, x)$, so

$$B_j(u, \Theta *'_j v) = d\,h(u, v) - h(\Theta u, v) = (d - j)h(u, v) + h(u, \Theta v) = B_j\big(u, (d - j) *'_j v + *'_j \Theta v\big)$$

for all u, using (1.1); since B_j is perfect, this is the star law. □

Thus, as bare existence statements, AW4-V and AW4-E in degree j are equivalent, and both are equivalent to "Θ_j is semisimple with all eigenvalues on $\mathrm{Re}\, s = j/2$"; in presence of the derivation rule and AW3, AW4-V in every degree is equivalent to AW4 (Proposition 6.1 and (iii)). The two forms differ in what a construction must supply: a form on the whole of V^j, or a positive involution on an algebra of operators containing the flow generator. Over $\mathbb{F}_q$ the second is what is actually proved for curves (Proposition 4.11 and Proposition 4.12).

Theorem 6.3 (Deninger's formal argument, after Serre; [Den94, (7.11)–(7.13), pp. 32–33], [Den98, p. 169], [Den10, pp. 4–5], [Den23, 2.7, p. 3])**.** *Let K be a number field and suppose that the single object $\overline{X}_K$ carries a Weil datum of type $(1, 0)$ satisfying AW1, AW2, and AW3 for $(\Lambda_K, \mathbb{1})$, and satisfying either AW4, or AW4-V in degree one, or AW4-E in degree one. Then:*

(i) $\mathrm{Sp}_1 = \mathrm{div}\, \xi_K$, *where* $\xi_K(s) = s(s - 1)\Lambda_K(s)$;
(ii) every zero of ξ_K *has real part* $\frac{1}{2}$;
(iii) Θ_1 *is diagonalizable, with* $\dim \ker(\Theta_1 - \rho) = \mathrm{ord}_\rho\, \xi_K$ *for every* $\rho \in \mathbb{C}$.

Proof of (i). By AW1 with $U = \mathbb{1}$, $(V^0, \Theta_0) = (\mathbb{C}, 0)$ and $(V^2, \Theta_2) \cong (\mathbb{C}, 1)$, so $\mathrm{Sp}_0 = [0]$ and $\mathrm{Sp}_2 = [1]$. By Theorem 3.5, Λ_K has simple poles at 0 and 1 and no others, so $\mathrm{div}\, \xi_K = \mathrm{div}\, \Lambda_K + [0] + [1]$. With $d = 1$, (AW-div) reads $\mathrm{div}\, \Lambda_K = -[0] + \mathrm{Sp}_1 - [1]$. □

Proof of (ii) and (iii). Under AW4, Proposition 6.1 (with $d = 1$, $w = 0$, $j = 1$) shows that h_1 is a positive infinitesimal polarization of weight one; under AW4-E in degree one, Proposition 6.2(i) provides one; under AW4-V it is given. Proposition 3.3 then gives $\mathrm{Re}\, \rho = \frac{1}{2}$ for every eigenvalue ρ of Θ_1 and $V^1_{\Theta\sim\rho} = \ker(\Theta_1 - \rho)$. By (i) the zeros of ξ_K, with multiplicity, are the eigenvalues of Θ_1, with multiplicity $m_1(\rho) = \dim \ker(\Theta_1 - \rho)$. □

By Proposition 3.6, every zero of ζ_K with $0 < \mathrm{Re}\, s < 1$ is a zero of ξ_K of the same order, so the conclusion contains the Riemann hypothesis for ζ_K; for $K = \mathbb{Q}$, where $\xi_{\mathbb{Q}} = 2\xi$, it is the Riemann hypothesis. The argument does not presuppose that zeros are simple, uses no regularized determinant and neither AW5 nor AW6, and uses from AW3 only the trace equivariance. In particular the normalization term of (AW-det) never enters: AW2 in the form (AW-div) identifies the zeros with the degree-one spectrum independently of any determinant convention. The statement is conditional: no structure satisfying AW1–AW4 on $\overline{X}_{\mathbb{Q}}$ is known.

Theorem 6.3 is Deninger's formal implication [Den94, (7.11)–(7.13), pp. 32–33], [Den98, pp. 168–169], [Den10, pp. 4–5], including the semisimplicity in (iii) [Den23, 2.7, p. 3], modelled

on Serre's argument [Ser60]; in Flach–Morin's formulation it is [FM20, Conjecture 2.3(iv)–(v), p. 522]. The statement here differs only in bookkeeping: AW2 is imposed in divisor form, the poles at 0 and 1 are carried by the endpoint clause of AW1 rather than by distinct eigenvalues in degrees 0, 1, 2 [Den98, p. 168], and AW4 may be replaced by either of its two other forms.

Definition 6.4. Let $\mathscr{H}$ be a Hilbert space with an orthonormal basis (e_k) and let (τ_k) be real numbers. The *diagonal operator* $T = \operatorname{diag}(\tau_k)$ has domain $\mathcal{D}(T) = \{\sum c_k e_k : \sum |\tau_k c_k|^2 < \infty\}$ and acts by $T(\sum c_k e_k) = \sum \tau_k c_k e_k$.

Proposition 6.5. *Let (V, Θ) be a spectral core, $j \in \mathbb{R}$, and h a positive infinitesimal polarization of weight j on (V, Θ). Let $\mathscr{H}$ be the Hilbert-space completion of (V, h). Then there is an orthonormal basis (e_k) of $\mathscr{H}$ contained in V with $\Theta e_k = (j/2 + i\tau_k)e_k$, $\tau_k \in \mathbb{R}$, and the diagonal operator $T = \operatorname{diag}(\tau_k)$ of Definition 6.4 is self-adjoint, satisfies $\Theta = j/2 + iT$ on V, and has pure point spectrum $\{\tau_k\}$ with finite multiplicities and no finite accumulation.*

Proof. By Proposition 3.3, each $V_{\Theta\sim\rho}$ is an eigenspace with $\rho = j/2 + i\tau$, and distinct eigenspaces are orthogonal. Choosing an orthonormal basis of each (finite-dimensional) eigenspace gives an orthonormal family (e_k) spanning V, hence an orthonormal basis of $\mathscr{H}$. The operator T is symmetric on its domain. If $y = \sum y_k e_k \in \mathcal{D}(T^*)$ with $T^* y = z$, then testing against $x = e_k$ gives $z_k = \tau_k y_k$, so $\sum |\tau_k y_k|^2 = \|z\|^2 < \infty$, $y \in \mathcal{D}(T)$, and $Ty = z$; hence $T = T^*$. The set $\{\tau_k\} = \{(\rho - j/2)/i : \rho \in \operatorname{spec}\Theta\}$ has no finite accumulation point because $\operatorname{spec}\Theta$ has none, so it is closed and discrete. The spectrum of a diagonal operator with real entries is the closure of the set of entries, which is therefore $\{\tau_k\}$, and each eigenspace is spanned by the finitely many e_k with that entry. □

Proposition 6.5 produces the diagonal completion, the Hilbert space envisaged in [Den94, p. 33] and [Den98, p. 169]. Axiom AW5 asks more: that a geometrically defined completion, with a geometrically defined unitary flow, coincides with it, with a skew-adjoint generator. A self-adjoint realization cannot be inferred from skew-symmetry of $\Theta_j - j/2$ on a dense domain alone (Proposition 9.6).

Proposition 6.6. *The following are equivalent.*

(i) The Riemann hypothesis holds.

(ii) There is a spectral core (V, Θ) with a positive infinitesimal polarization of weight one such that $\operatorname{spec}\Theta$ with multiplicities is the multiset of nontrivial zeros of ζ.

Proof of (i)⇒(ii). Let $V = \bigoplus_\rho \mathbb{C}^{m(\rho)}$ over the distinct nontrivial zeros ρ, with $m(\rho)$ the order of the zero, $\Theta = \rho$ on the ρ-summand, and h the orthogonal sum of the standard forms. The zeros are discrete (they are zeros of the entire function ξ), so (V, Θ) is a spectral core. For u, v in the ρ-summand, $h(\Theta u, v) + h(u, \Theta v) = (\rho + \bar\rho)h(u, v) = h(u, v)$ because $\operatorname{Re}\rho = \frac{1}{2}$; distinct summands are orthogonal on both sides. □

Proof of (ii)⇒(i). Proposition 3.3(i). □

Remark 6.7. Proposition 6.6 shows that the bare existence of a positive spectral model is a reformulation of RH. The model in its proof supplies no cup product, trace, functoriality, local comparison, or trace formula, and it does not establish the determinant identity (AW-det) for the diagonal operator. The research problem is to derive the spectrum and the positivity from an independently defined arithmetic object; compare Propositions 4.2 and 4.3 over $\mathbb{F}_q$. The idea of realizing the zeros as a spectrum goes back to Hilbert and Pólya; see [Con99, p. 3], [Den92, p. 147], and [Sar04, p. 5].

6.1. **Positivity cannot be weakened.** The first two negative findings (R1 and R2 of Section 9) are stated here, before the main theorem, because Section 7.3 uses them: they show that Proposition 3.3 fails as soon as its form is only nondegenerate, or positive on part of the space.

It is natural to expect that duality for a flow, together with a Hermitian form compatible with it, forces purity, and that positivity of such a form on one spectral piece (for instance of a height pairing on one eigenspace) propagates to all of them; in fact neither holds without positivity on the whole space.

Definition 6.8. Let $j \in \mathbb{R}$ and let $V = \mathbb{C}^2$ with standard basis e_1, e_2.

(a) For $\rho \in \mathbb{C}$, the *dual pair* is $\Theta_\rho^{\mathrm{dual}} = \mathrm{diag}(\rho,\, j-\bar\rho)$ with the form $h^{\mathrm{dual}}(u,v) = u_1\bar v_2 + u_2\bar v_1$.

(b) For real $\tau_1 \neq \tau_2$, the *split pair* is $\Theta^{\mathrm{split}} = \mathrm{diag}(j/2 + i\tau_1,\, j/2 + i\tau_2)$ with the form $h^{\mathrm{split}}(u,v) = u_1\bar v_1 - u_2\bar v_2$.

Proposition 6.9. *Let $j \in \mathbb{R}$ and let $V = \mathbb{C}^2$ with standard basis e_1, e_2.*

(i) For $\rho \in \mathbb{C}$, h^{dual} is a nondegenerate Hermitian form satisfying (1.1) for $\Theta = \Theta_\rho^{\mathrm{dual}}$, and $\Theta_\rho^{\mathrm{dual}}$ has the eigenvalue ρ; in particular $\mathrm{Re}\,\rho \neq j/2$ is possible.

(ii) For real $\tau_1 \neq \tau_2$, h^{split} is a nondegenerate Hermitian form satisfying (1.1) for $\Theta = \Theta^{\mathrm{split}}$, it is positive definite on the eigenline $\ker(\Theta^{\mathrm{split}} - j/2 - i\tau_1)$, and $h^{\mathrm{split}}(e_2,e_2) = -1$.

Proof of (i). Put $\Theta = \Theta_\rho^{\mathrm{dual}}$ and $h = h^{\mathrm{dual}}$, and write $h(u,v) = u^T H \bar v$ with $H = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$; H is real symmetric and invertible, so h is Hermitian and nondegenerate. Identity (1.1) is equivalent to $\Theta^T H + H\bar\Theta = jH$. Both sides vanish on the diagonal, and the off-diagonal entries of the left side are $\rho + \overline{(j-\bar\rho)} = j$ and $(j-\bar\rho) + \bar\rho = j$. □

Proof of (ii). Put $\Theta = \Theta^{\mathrm{split}}$ and $h = h^{\mathrm{split}}$. Here $H = \mathrm{diag}(1,-1)$, and $\Theta^T H + H\bar\Theta$ is diagonal with entries $2\,\mathrm{Re}(j/2 + i\tau_k)(\pm 1) = j(\pm 1)$, so (1.1) holds. The eigenline for $j/2 + i\tau_1$ is $\mathbb{C}e_1$, on which $h(ce_1, ce_1) = |c|^2$; and $h(e_2,e_2) = -1$. □

In (i) both eigenlines are h-isotropic, as they must be by the computation in the proof of Proposition 3.3(i); positivity, not nondegeneracy, is what excludes this. Item (ii) is the orthogonal sum of a positive and a negative line. Deninger's comparison with heights [Den10, pp. 7–9] is therefore motivation and a compatibility test, not a construction of the full positive form required by AW4. Proposition 6.11 below shows that the gap persists in presence of a cup product, a trace, and a quaternionic star.

It is natural to expect that a quaternionic star, perfect weight-one duality, and the star law of AW4 together force eigenvalues onto the critical line; in fact all of this structure is compatible with eigenvalues off the line, the form h_1 then being indefinite.

Definition 6.10. Let $0 < a < \frac12$, $T > 0$, and $\lambda = \frac12 + a + iT$. Let $V = \mathbb{C}^4$ with standard basis e_1, e_2, f_1, f_2 (in this order), let I_2 be the 2×2 identity matrix, and put

$$\Theta = \mathrm{diag}(\lambda, \bar\lambda, 1-\lambda, 1-\bar\lambda), \quad \Omega = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}.$$

Let $B(v,w) = v^T\Omega w$ and let $J : V \to V$ be the antilinear map $Jv = S\bar v$. The *quaternionic block* is the graded algebra $V^0 = \mathbb{C}\cdot 1$, $V^1 = V$, $V^2 = \mathbb{C}\eta$ with unit 1, products $v \cup w = B(v,w)\eta$ for $v, w \in V^1$ and all other products of positive-degree elements zero, generators $\Theta_0 = 0$, $\Theta_1 = \Theta$, $\Theta_2 = 1$, trace $\mathrm{Tr}(c\eta) = c$, and antilinear stars $*_0(c\cdot 1) = \bar c\eta$, $*_1 = J$, $*_2(c\eta) = \bar c\cdot 1$. Put $h_j(u,v) = \mathrm{Tr}(u \cup *_j v)$. We also use the antilinear map $J_+ v = -\Omega\bar v$ on V, the matrix $\Pi = \left(\begin{smallmatrix} 0 & I_2 \\ I_2 & 0 \end{smallmatrix}\right)$, the characteristic polynomial $p(s) = \det(s - \Theta)$, and the variable $w = s - \frac12$.

Proposition 6.11. *Let $0 < a < \frac12$ and $T > 0$, and let the quaternionic block of Definition 6.10 be given. Then:*

(i) *the block, regarded as a Weil datum of type* $(1,0)$ *(Definition 1.4), satisfies AW1 for* $U = \mathbb{H}$*, AW3, and the star law of AW4 in degrees* $0,1,2$*; moreover* $J^2 = -1$*, and* h_0, h_2 *are positive definite;*
(ii) $h_1(v,w) = v^T H \bar{w}$ *with* $H = \Omega S = \left(\begin{smallmatrix} 0 & R \\ -R & 0 \end{smallmatrix}\right)$*,* h_1 *is Hermitian and nondegenerate of signature* $(2,2)$*, and* $h_1(e_1+f_2, e_1+f_2) = -2$*;*
(iii) *every eigenvalue of* Θ_1 *has real part* $\frac{1}{2}+a$ *or* $\frac{1}{2}-a$*, hence none lies on* $\operatorname{Re} s = \frac{1}{2}$*.*

Proof of (i). Since Θ is diagonal and the only nonzero entries of Ω are at positions $(1,3),(2,4)$ (value 1) and $(3,1),(4,2)$ (value -1), the identity $\Theta^T\Omega + \Omega\Theta = \Omega$ reduces to $\lambda + (1-\lambda) = 1$ and $\bar{\lambda} + (1-\bar{\lambda}) = 1$. Hence $B(\Theta v, w) + B(v, \Theta w) = B(v,w)$, which is the derivation rule in degrees $1+1$; in all other bidegrees the derivation rule reads $0 = 0$ or involves the unit. The form B is alternating because $\Omega^T = -\Omega$, so the product is graded-commutative; it is associative because every product of three positive-degree elements lies in degree at least 3, where the algebra vanishes. The normalizations $V^0 = \mathbb{C}\cdot 1$, $\Theta_0 = 0$, $V^2 = \mathbb{C}$, $\Theta_2 = 1$ hold by definition. For AW3, $\operatorname{Tr}(\Theta_2 x) = \operatorname{Tr}(x)$; the pairing $V^0 \times V^2 \to \mathbb{C}$ is $(c, c'\eta) \mapsto cc'$; and on V^1 the eigenlines are $\mathbb{C}e_1, \mathbb{C}e_2, \mathbb{C}f_1, \mathbb{C}f_2$ with eigenvalues $\lambda, \bar{\lambda}, 1-\lambda, 1-\bar{\lambda}$, and $B(e_1,f_1) = B(e_2,f_2) = 1$, so the eigenline for ρ is paired perfectly with that for $1-\rho$. For the star law in degree one, $J\Theta = \Theta J$ amounts to $\Theta S = S\bar{\Theta}$; since S interchanges the first two and the last two coordinates up to sign, this reduces to $\lambda = \overline{\bar{\lambda}}$ and $1-\lambda = \overline{1-\bar{\lambda}}$. In degree zero, $\Theta_2 *_0 = *_0 \Theta_0 + *_0$ holds since both sides send c to $\bar{c}\eta$; in degree two, $\Theta_0 *_2 = 0 = *_2\Theta_2 - *_2$. Since S is real and $S^2 = -I$, $J^2 v = S\overline{S\bar{v}} = S^2 v = -v$. Finally $h_0(c,c') = h_2(c\eta, c'\eta) = c\bar{c}'$. □

Proof of (ii). $h_1(v,w) = B(v, Jw) = v^T \Omega S \bar{w}$, and a direct multiplication gives $\Omega S = \left(\begin{smallmatrix} 0 & R \\ -R & 0 \end{smallmatrix}\right)$. Since $R^T = -R$, H is real symmetric, so h_1 is Hermitian. Moreover $H^2 = \operatorname{diag}(-R^2, -R^2) = I$ and $\operatorname{tr} H = 0$, so the eigenvalues of H are 1 and -1, each with multiplicity two. With $v = e_1 + f_2 = (1,0,0,1)$, $v^T H v = 2H_{14} = 2R_{12} = -2$. □

Proof of (iii). The eigenvalues are $\frac{1}{2} \pm a \pm iT$ and $0 < a$. □

By Proposition 6.1, h_1 satisfies (1.1) with $j = 1$; Proposition 3.3 does not apply because h_1 is indefinite, and indeed every eigenline is h_1-isotropic (H has zero diagonal). Axiom AW2 is not addressed: the block is a witness against deducing purity from AW1, AW3, and the algebraic part of AW4, not a model of all axioms.

It is natural to expect that, if the given star is not positive, one may choose a positive star afterwards and keep the flow compatibility; in fact positivity and compatibility with the flow must hold for the same star.

Proposition 6.12. *Let* $0 < a < \frac{1}{2}$ *and* $T > 0$*, and let* $V, \Theta, \Omega, B, J_+, \Pi$ *be as in Definition 6.10. Then:*

(i) $J_+^2 = -1$ *and* $B(v, J_+ w) = v^T \bar{w}$*, a positive definite Hermitian form;*
(ii) $(\Theta J_+ - J_+\Theta)v = -2a\,\Pi\bar{v}$ *for all* $v \in V$*; in particular* $\Theta J_+ \neq J_+\Theta$*;*
(iii) *there is no antilinear map* $J' : V \to V$ *with* $\Theta J' = J'\Theta$ *for which* $(v,w) \mapsto B(v, J'w)$ *is a positive definite Hermitian form.*

Proof of (i). $J_+^2 v = -\Omega\overline{(-\Omega\bar{v})} = \Omega^2 v = -v$, and $B(v, J_+ w) = -v^T\Omega^2\bar{w} = v^T\bar{w}$. □

Proof of (ii). $\Theta J_+ v - J_+\Theta v = -(\Theta\Omega - \Omega\bar{\Theta})\bar{v}$. The matrix $\Theta\Omega - \Omega\bar{\Theta}$ is supported on the nonzero positions of Ω, with entries $\lambda - (1-\bar{\lambda})$ at $(1,3)$, $\bar{\lambda} - (1-\lambda)$ at $(2,4)$, $-(1-\lambda) + \bar{\lambda}$ at $(3,1)$, and $-(1-\bar{\lambda}) + \lambda$ at $(4,2)$; each equals $\lambda + \bar{\lambda} - 1 = 2a$. Hence $\Theta\Omega - \Omega\bar{\Theta} = 2a\Pi$, which is nonzero since $a > 0$. □

Proof of (iii). Suppose J' is such a map and put $h'(v,w) = B(v, J'w)$. Using $\Theta J' = J'\Theta$ and $\Theta^T\Omega + \Omega\Theta = \Omega$ (proof of Proposition 6.11(i)),

$$h'(\Theta v, w) + h'(v, \Theta w) = B(\Theta v, J'w) + B(v, \Theta J'w) = B(v, J'w) = h'(v,w).$$

Proposition 3.3(i) with $j = 1$ then gives $\operatorname{Re}\rho = \frac{1}{2}$ for every eigenvalue, contradicting Proposition 6.11(iii). □

The commutation defect in (ii) has operator norm $2a$, twice the distance of the eigenvalues from the critical line.

7. The class of zeta functions and the main theorem

This section states the main theorem, for the class

$$\mathcal{C}_{\mathrm{Ded}} = \{\Lambda_K : K \text{ a number field}\}$$

of completed Dedekind zeta functions, which contains $\Lambda = \Lambda_{\mathbb{Q}}$. Its objects $\overline{X}_K$ are the arithmetic curves, so the finite-field theorem it mirrors is the Riemann hypothesis for curves over $\mathbb{F}_q$ (Theorem 7.24); its analytic inputs (continuation, functional equation, gamma factors) are theorems; and the Artin formalism expresses finite-order Hecke and Artin L-functions through it, so the theorem also covers their zeros in the critical strip. The section then isolates the axiom that carries the content (Section 7.3), treats Galois descent (Section 7.4), places the finite-field theorem in the same frame (Section 7.5), and records the statement for higher-dimensional arithmetic schemes (Section 7.6).

7.1. Artin and Hecke L-functions.

To avoid a clash with the zeros ρ and the constant π, Galois representations are denoted σ.

Definition 7.1. Let k be a number field, L/k a finite Galois extension with group G, and $\sigma : G \to \mathrm{GL}(V_\sigma)$ a representation on a finite-dimensional complex vector space.

(a) The *Artin L-function* $L(\sigma, s)$ is the Euler product $\prod_{\mathfrak{p}} \det\bigl(1 - \sigma(\mathrm{Frob}_{\mathfrak{P}})N\mathfrak{p}^{-s} \mid V_\sigma^{I_{\mathfrak{P}}}\bigr)^{-1}$ over the finite primes $\mathfrak{p}$ of k, where $\mathfrak{P}$ is a prime of L above $\mathfrak{p}$ with inertia group $I_{\mathfrak{P}}$, for $\operatorname{Re} s > 1$ [Cog14, pp. 9 and 11–12].

(b) The *completed Artin L-function* is $\Lambda(\sigma, s) = L_\infty(\sigma, s)L(\sigma, s)$, where $L_\infty(\sigma, s)$ is the product over the archimedean places v of k of $\Gamma_{\mathbb{R}}(s)^{n_+}\Gamma_{\mathbb{R}}(s+1)^{n_-}$ if v is real and σ restricted to a decomposition group at v is $n_+ \cdot 1 \oplus n_- \cdot \mathrm{sgn}$, and of $\Gamma_{\mathbb{C}}(s)^{\dim\sigma}$ if v is complex [Cog14, p. 13].

(c) The *dual* and the *conjugate* of σ are $\sigma^\vee$ and $\bar\sigma$; an irreducible σ is *self-dual* if $\sigma \cong \sigma^\vee$, and *defined over* $\mathbb{R}$ if $V_\sigma \cong V_0 \otimes_{\mathbb{R}} \mathbb{C}$ for a real representation V_0 of G.

(d) For a modulus $\mathfrak{m}$ of k and a character χ of the ray class group of $\mathfrak{m}$, the *finite-order Hecke L-function* is the Weber L-series $L(\chi, s) = \prod_{\mathfrak{p}\nmid\mathfrak{m}}(1 - \chi(\mathfrak{p})N\mathfrak{p}^{-s})^{-1}$ [Mil20, p. 8].

For $\sigma = 1_G$, $\Lambda(1_G, s) = \Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}\zeta_k(s) = |D_k|^{-s/2}\Lambda_k(s)$, so $\Lambda(1_G, s)$ and Λ_k have the same divisor.

Theorem 7.2 (Artin; Hecke; Brauer)**.** *Let k, L/k, and G be as in Definition 7.1.*

(i) The functions $\Lambda(\sigma, s)$ are additive in σ, unchanged by inflation from a quotient $\mathrm{Gal}(L'/k)$ of G, and inductive: for $H \subset G$ and a representation χ of $H = \mathrm{Gal}(L/L^H)$, $\Lambda(\mathrm{Ind}_H^G\chi, s) = \Lambda(\chi, s)$, the right side formed over L^H [Cog14, pp. 9, 12, and 13].

(ii) Each $\Lambda(\sigma, s)$ extends meromorphically to $\mathbb{C}$ and satisfies $\Lambda(\sigma, s) = \varepsilon(\sigma, s)\Lambda(\sigma^\vee, 1-s)$, where $\varepsilon(\sigma, s) = W(\sigma)A_\sigma^{-(s-1/2)}$ with $|W(\sigma)| = 1$ and a real number $A_\sigma > 0$ [Cog14, pp. 10–11, 13, and 15].

(iii) *(Brauer)* $L(\sigma,s)=\prod_i L(\psi_i,s)^{n_i}$ *with* $n_i\in\mathbb{Z}$, *where* ψ_i *are one-dimensional characters of subgroups* H_i *of* G *and* $L(\psi_i,s)$ *is formed over* L^{H_i} [Cog14, pp. 10–11, (2)], [Kra05, Theorem 2, p. 7].

(iv) *(Artin reciprocity) The Artin* L*-function of a one-dimensional representation of* $\mathrm{Gal}(L/k)$ *is, up to finitely many Euler factors, a finite-order Hecke* L*-function of* k [Mil20, p. 10].

(v) *(Weber, Hecke) A finite-order Hecke* L*-function* $L(\chi,s)$ *with* χ *nontrivial extends to an entire function* [Mil20, p. 8]*; and if* M *is the class field of a ray class group of* k*, then up to finitely many Euler factors* $\zeta_M(s)=\prod_\chi L(\chi,s)$*, the product over the characters* χ *of that ray class group* [Cog14, p. 4].

The sources cited state these results; the proofs are in the works of Artin, Hecke, and Brauer to which they refer, except that [Kra05] proves the group-theoretic part of (iii). An Euler factor at a finite prime has zeros and poles only on $\operatorname{Re}s=0$, so the finitely many factors in (iv) and (v) do not affect zeros or poles in $0<\operatorname{Re}s<1$. Two consequences are used repeatedly. First, by (i) and the decomposition of $\mathrm{Ind}_H^G 1$ into irreducibles,

$$\operatorname{div}\Lambda_{L^H}=\sum_\sigma \langle\sigma,\mathrm{Ind}_H^G 1\rangle\operatorname{div}\Lambda(\sigma,s), \tag{7.1}$$

the sum over the irreducible representations of G, where $\langle\,,\rangle$ is the multiplicity pairing of characters. Second, by (ii), $\operatorname{ord}_{1-\rho}\Lambda(\sigma^\vee,s)=\operatorname{ord}_\rho\Lambda(\sigma,s)$ for all ρ. If σ is self-dual, applying (ii) twice gives $W(\sigma)^2=1$, so $W(\sigma)=\pm1$, and the function $t\mapsto A_\sigma^{t/2}\Lambda(\sigma,\frac12+t)$ is even if $W(\sigma)=1$ and odd if $W(\sigma)=-1$.

Theorem 7.3 (Frobenius–Schur; Fröhlich–Queyrut; Serre)**.** *Let* G *be a finite group and* σ *an irreducible complex representation of* G.

(i) *If* σ *is self-dual, then up to scaling there is exactly one nonzero* G*-invariant bilinear form on* V_σ*; it is nondegenerate and either symmetric or alternating, and it is symmetric if and only if* σ *is defined over* $\mathbb{R}$ [Con, Theorem 2.1, p. 2].

(ii) *If* $G=\mathrm{Gal}(L/k)$ *for number fields and* σ *is defined over* $\mathbb{R}$*, then* $W(\sigma)=1$ *(Fröhlich and Queyrut, as quoted in* [Den23, p. 6]*).*

(iii) *If* σ *is self-dual and not defined over* $\mathbb{R}$*, it occurs with even multiplicity in the complexification of every real representation of* G [Ser77, §13.2]*, as quoted in* [Den23, p. 5].

(iv) *There is a Galois extension* $L/\mathbb{Q}$ *with group the quaternion group of order eight whose two-dimensional irreducible representation* σ *has* $W(\sigma)=-1$ *(constructed by Armitage, as reported in* [Cou83, §4, p. 29]*; examples due to Omar are quoted in* [Den23, p. 5]*).*

The two-dimensional irreducible representation of the quaternion group has character with values $2,-2,0$, which are rational, and is self-dual and not defined over $\mathbb{R}$ [Con, Example 1.1, p. 1].

7.2. **The main theorem.**

Theorem 7.4 (Theorem A; (i) and (iv) after Deninger, (ii) and (iii) by the Artin formalism)**.** *Suppose that for every number field* K *the object* $\overline{X}_K$ *carries a Weil datum of type* $(1,0)$ *satisfying AW1, AW2, AW3, and AW4 for* $(\Lambda_K,\nVdash)$*. Then:*

(i) *for every number field* K*, every zero of* ξ_K*, that is every nontrivial zero of* ζ_K*, has real part* $\frac12$*; in particular the Riemann hypothesis holds;*

(ii) *every zero in* $0<\operatorname{Re}s<1$ *of every finite-order Hecke* L*-function of every number field has real part* $\frac12$*;*

(iii) *every zero and every pole in* $0<\operatorname{Re}s<1$ *of every Artin* L*-function* $L(\sigma,s)$*, for every finite Galois extension of number fields and every representation* σ*, has real part* $\frac12$*;*

(iv) for every K, Θ_1 is diagonalizable on V_K^1, with $\dim\ker(\Theta_1-\rho)=\operatorname{ord}_\rho\zeta_K$ for every nontrivial zero ρ of ζ_K and $\ker(\Theta_1-\rho)=0$ otherwise, and $\Theta_1-\frac12$ is skew-Hermitian for h_1.

The hypothesis is used object by object: (i) and (iv) for a given K need the axioms only on $\overline{X}_K$, and (ii)–(iii) only on the objects $\overline{X}_M$ for the fields M cut out by the characters involved; no functoriality is used. By Theorem 6.3, AW4 may be replaced by AW4-V or AW4-E in degree one; the conclusions remain valid, except that in the last clause of (iv) the form h_1 is replaced by the polarization provided by that hypothesis.

Proof of (i) and (iv). Theorem 6.3 gives both, except for the orders and the last clause. For ρ with $0<\operatorname{Re}\rho<1$ the order of ξ_K equals that of ζ_K (Proposition 3.6), and by Theorem 6.3(ii) every zero of ξ_K lies in that strip. By Proposition 6.1, h_1 is a positive infinitesimal polarization of weight one, which says that $\Theta_1-\frac12$ is skew-Hermitian for h_1. □

Proof of (ii). Let χ be a character of the ray class group of a modulus of k, M the corresponding class field, and ρ a zero of $L(\chi,s)$ with $0<\operatorname{Re}\rho<1$. By Theorem 7.2(v), up to finitely many Euler factors, which have no zeros or poles in the strip, $\zeta_M(s)=\prod_{\chi'}L(\chi',s)$. Each factor with $\chi'\neq1$ is entire, and the factor with $\chi'=1$ agrees with ζ_k up to finitely many Euler factors, so it is holomorphic in the strip by Theorem 3.5. Hence $\operatorname{ord}_\rho\zeta_M\geq\operatorname{ord}_\rho L(\chi,s)>0$; by Proposition 3.6, ρ is a zero of ξ_M, and $\operatorname{Re}\rho=\frac12$ by (i) for M. □

Proof of (iii). Let σ be a representation of $G=\mathrm{Gal}(L/k)$. By Theorem 7.2(iii), $L(\sigma,s)=\prod_i L(\psi_i,s)^{n_i}$ with $n_i\in\mathbb{Z}$, and by Theorem 7.2(iv) each $L(\psi_i,s)$ is, up to finitely many Euler factors, a finite-order Hecke L-function of L^{H_i}. As in the previous proof each $L(\psi_i,s)$ is holomorphic on $0<\operatorname{Re}s<1$. If ρ in that strip is a zero or pole of $L(\sigma,s)$, then $0\neq\operatorname{ord}_\rho L(\sigma,s)=\sum_i n_i\operatorname{ord}_\rho L(\psi_i,s)$, so some $L(\psi_i,s)$ vanishes at ρ, and $\operatorname{Re}\rho=\frac12$ by (ii). □

Items (i) and (iv) are Deninger's implication (Theorem 6.3 and the discussion after it); items (ii) and (iii) are the classical consequence of the Riemann hypothesis for all ζ_K through Brauer induction, Artin reciprocity, and Hecke's theory (Theorem 7.2). Nothing is gained for zero location by enlarging the class to Hecke or Artin L-functions: $\mathcal{C}_{\mathrm{Ded}}$ already controls them through the Artin formalism. What functoriality adds is multiplicity information and Artin's holomorphy conjecture for the rational orbits of characters (Proposition 7.12). Subsection 14.13 records which parts of Theorem 7.4 are formalized.

7.3. **Which axiom carries the content.** It is natural to expect that a Weil datum with the correct spectrum and perfect duality is already a nontrivial arithmetic statement, and that a flow-compatible Hermitian form, or a positive one, is what forces purity; in fact AW1–AW3 can be satisfied with any symmetric spectrum, each half of AW4 can be satisfied separately with zeros off the line, and the full package on one object is equivalent to the Riemann hypothesis for that object. The same is true over $\mathbb{F}_q$: a positive Frobenius-compatible form exists if and only if Frobenius is semisimple and pure (Proposition 4.3), and the content of Weil's theorem lies in constructing the form from geometry.

Definition 7.5. Let $\Sigma=\sum_\rho m(\rho)[\rho]$ be a divisor on $\mathbb{C}$ with $m(\rho)\geq0$, no finite accumulation point of its support, invariant under $\rho\mapsto1-\rho$, and with $m(\frac12)$ even. Choose

(a) for each unordered pair $\{\rho,1-\rho\}$ with $\rho\neq\frac12$ in the support, bases $(e_{\rho,k})_{k\leq m(\rho)}$ and $(e_{1-\rho,k})_{k\leq m(\rho)}$, one of them designated first;
(b) a basis $(e_{1/2,k})_{k\leq m(1/2)}$ grouped into consecutive pairs.

The *trivial model* with spectrum Σ is the Weil datum of type $(1,0)$ with $V^0 = \mathbb{C}\cdot 1$, $V^2 = \mathbb{C}\eta$, V^1 the algebraic direct sum of the lines $\mathbb{C}e_{\rho,k}$, generators $\Theta_0 = 0$, $\Theta_2 = 1$, $\Theta_1 e_{\rho,k} = \rho e_{\rho,k}$, the alternating form B on V^1 with $B(e,f) = 1 = -B(f,e)$ for each pair (e,f) (the first member of a pair in (a) being the designated vector, the pairs in (b) in their given order) and $B = 0$ on all other pairs of basis vectors, products $u \cup v = B(u,v)\eta$ on V^1 and the unit products otherwise, trace $\mathrm{Tr}(c\eta) = c$, and stars $*_0(c) = \bar{c}\eta$, $*_2(c\eta) = \bar{c}$, and $*_1$ the antilinear map with $*_1 e = f$, $*_1 f = -e$ on each pair (e,f) if Σ is supported on $\mathrm{Re}\, s = \frac{1}{2}$, and $*_1 = 0$ otherwise.

Proposition 7.6. *Let Σ be as in Definition 7.5 and consider its trivial model.*

(i) It satisfies AW1 for $U = \mathbb{H}$ and AW3, and $\mathrm{Sp}_1 = \Sigma$.
(ii) If Σ is supported on $\mathrm{Re}\, s = \frac{1}{2}$, it satisfies AW4, and $*_1^2 = -1$.
(iii) Some Weil datum with $\mathrm{Sp}_1 = \Sigma$ satisfies AW1 for $U = \mathbb{H}$, AW3, and AW4 if and only if Σ is supported on $\mathrm{Re}\, s = \frac{1}{2}$.

Proof of (i). The spectrum of Θ_1 is Σ by construction and has no finite accumulation point, so each (V^j, Θ_j) is a spectral core. The product is associative (every product of three positive-degree elements lies in degree at least three, where it vanishes) and graded-commutative (B is alternating). The derivation rule is $0 = 0$ in the bidegrees involving V^0 trivially, and in bidegree $(1,1)$ it reads $B(\Theta u, v) + B(u, \Theta v) = B(u,v)$, which holds on basis vectors because $B(e_{\rho,k}, e_{\rho',l}) \neq 0$ forces $\rho + \rho' = 1$. The endpoint conditions hold by construction. For AW3, $\mathrm{Tr}(\Theta_2 x) = \mathrm{Tr}(x)$; the pairing $V^0 \times V^2$ is $(c, c'\eta) \mapsto cc'$; and B pairs the span of the $e_{\rho,k}$ perfectly with the span of the $e_{1-\rho,k}$. □

Proof of (ii). On the line, $1 - \rho = \bar{\rho}$, so $*_1$ maps the ρ-eigenspace to the $\bar{\rho}$-eigenspace and commutes with Θ_1 (for an antilinear map this is $\Theta_1 *_1 e = \bar{\rho} *_1 e = *_1(\rho e)$), which is the star law with $d = j = 1$. The laws in degrees 0 and 2 are $\Theta_2 *_0 = *_0 \Theta_0 + *_0$ and $\Theta_0 *_2 = *_2 \Theta_2 - *_2$, both immediate. On a pair (e,f), $h_1(e,e) = B(e,f) = 1$, $h_1(f,f) = B(f,-e) = 1$, $h_1(e,f) = B(e,-e) = 0$, and $h_1(f,e) = B(f,f) = 0$; distinct pairs are h_1-orthogonal. So h_1 is the standard positive form in the basis, and $h_0(c,c') = h_2(c\eta, c'\eta) = c\bar{c}'$. Finally $*_1^2 e = *_1 f = -e$ and $*_1^2 f = -*_1 e = -f$. □

Proof of (iii). If Σ lies on the line, use (i) and (ii). Conversely, by Proposition 6.1 the form h_1 is a positive infinitesimal polarization of weight one, and Proposition 3.3(i) puts every eigenvalue on the line. □

Proposition 7.7. *Let K be a number field and $\Sigma_K = \mathrm{div}\, \xi_K$.*

(i) The trivial model with spectrum Σ_K satisfies AW1, AW2, and AW3 for $(\Lambda_K, \mathbb{H})$, unconditionally.
(ii) The trivial model with spectrum Σ_K carries a positive definite Hermitian form on V^1 (not of the form h_1 and not satisfying (1.1) unless Σ_K lies on the line).
(iii) For every divisor Σ as in Definition 7.5 which is supported in $0 < \mathrm{Re}\, s < 1$ and invariant under $\rho \mapsto \bar{\rho}$ there is a Weil datum of type $(1,0)$ satisfying AW1 for $U = \mathbb{H}$, AW3, and the star law of AW4, with $\mathrm{Sp}_1 = \Sigma$ and with star form h_1 Hermitian, nondegenerate, and satisfying (1.1) with weight one; h_1 is indefinite as soon as Σ has a point off the line.
(iv) Some Weil datum on $\overline{X}_K$ satisfies AW1–AW4 for $(\Lambda_K, \mathbb{H})$ if and only if every zero of ξ_K has real part $\frac{1}{2}$.

Proof of (i). Σ_K is invariant under $\rho \mapsto 1 - \rho$ by Proposition 3.6, its support has no finite accumulation point, and $m(\frac{1}{2}) = \mathrm{ord}_{1/2}\, \xi_K$ is even because $t \mapsto \xi_K(\frac{1}{2} + t)$ is an even function. By Proposition 7.6(i) the trivial model satisfies AW1 and AW3 with $\mathrm{Sp}_1 = \Sigma_K$, and then (AW-div) holds exactly as in the proof of Theorem 6.3(i), read backwards. □

Proof of (ii). Take the form for which the basis vectors $e_{\rho,k}$ are orthonormal. If it satisfied (1.1), Proposition 3.3 would put Σ_K on the line. □

Proof of (iii). The support of Σ splits into the points on the line, the real pairs $\{\rho, 1-\rho\}$ with $\rho \neq \frac{1}{2}$, and the quartets $\{\lambda, \bar{\lambda}, 1-\lambda, 1-\bar{\lambda}\}$ with λ nonreal and $\operatorname{Re}\lambda \neq \frac{1}{2}$; the multiplicity is constant on each pair and quartet. Take $V^0 = \mathbb{C}\cdot 1$, $V^2 = \mathbb{C}\eta$ and let V^1 be the algebraic direct sum of the following blocks, with products $u \cup v = B(u,v)\eta$ for the direct sum B of the blocks' forms, and star the direct sum of the blocks' stars. For the part on the line, the trivial model of Definition 7.5, whose star form is positive (Proposition 7.6(ii)). For each real pair and each unit of multiplicity, the plane $\mathbb{C}e \oplus \mathbb{C}f$ with $\Theta = \operatorname{diag}(\rho, 1-\rho)$, $B(e,f) = 1 = -B(f,e)$, $B(e,e) = B(f,f) = 0$, and $*(ae+bf) = -\bar{a}e + \bar{b}f$; then $*$ commutes with the real diagonal Θ, and $h_1(ae+bf, a'e+b'f) = a\bar{b}'+b\bar{a}'$ is the dual-pair form of Definition 6.8 with $j=1$, of signature $(1,1)$. For each quartet and each unit of multiplicity, choose λ in it with $\operatorname{Re}\lambda > \frac{1}{2}$ and $\operatorname{Im}\lambda > 0$ and take the block V^1 of Definition 6.10 with $a = \operatorname{Re}\lambda - \frac{1}{2} \in (0, \frac{1}{2})$ and $T = \operatorname{Im}\lambda$, with its form B and star J; by Proposition 6.11 it satisfies the axioms listed and its star form has signature $(2,2)$. The direct sum satisfies AW1 and AW3 blockwise as in the proof of Proposition 7.6(i) (the form B pairs the ρ- and $(1-\rho)$-eigenspaces of each block), and the star law blockwise. By Proposition 6.1, h_1 satisfies (1.1). It is the orthogonal sum of the blocks' forms, hence nondegenerate, and indefinite if some block is off the line. □

Proof of (iv). If every zero of ξ_K lies on the line, the trivial model with spectrum Σ_K satisfies AW1–AW3 by (i) and AW4 by Proposition 7.6(ii). The converse is Theorem 6.3. □

Remark 7.8. Proposition 7.7 is the exact analogue, for one object, of Propositions 4.2 and 4.3 over $\mathbb{F}_q$: positivity *and* flow compatibility of one and the same star is the whole content of the axioms, and dropping either half destroys the implication ((ii) and (iii); for a positive star that fails to commute with the flow see also Proposition 6.12). The axioms are therefore to be judged not by the implication, which is formal, but by whether the functor, cup product, trace, descent, and trace formula can be constructed independently of the zeros (Section 13). The same collapse happens for any single meromorphic function G with simple poles at 0 and 1 and no other poles, with $\operatorname{div} G$ invariant under $\rho \mapsto 1-\rho$ and even central order, for instance a completed member of the Selberg class that is self-dual (so that $\operatorname{div} G$ is invariant under $\rho \mapsto 1-\rho$), has root number $+1$ (so that the central order is even), and has a simple pole at $s=1$: Proposition 7.6(iii), applied to the divisor of the entire function $s(s-1)G(s)$, shows that "a cohomology for G" of type $(1,0)$ exists if and only if the zeros of G lie on the line. (For entire G the endpoint blocks of AW1 would have to be removed, and for poles of higher order they do not suffice.) An analytic class supplies no object for such a cohomology to be a functor of, no cup product with an independent meaning, no descent, and no geometric side of a trace formula; this is why the headline class is geometric.

It is natural to expect that self-duality and a simple pole at $s=1$ suffice for the collapse of Remark 7.8, so that for such a function a "cohomology" of type $(1,0)$ exists if and only if its zeros lie on the line; in fact the even central order is a separate condition, and an odd central order excludes every Weil datum with the axioms other than positivity, whatever the location of the zeros.

Proposition 7.9 (after [Den23, 2.9, p. 4, and Theorem 2.13, p. 5])**.** *Let σ be the two-dimensional irreducible representation of the quaternion group of order eight, realized as the Galois group of an extension $L/\mathbb{Q}$ with $W(\sigma) = -1$ as in Theorem 7.3(iv), let $\Lambda(\sigma, s)$ be its completed Artin L-function over $\mathbb{Q}$, and put $G(s) = \Lambda(s)\Lambda(\sigma, s)$.*

(i) $\operatorname{div} G$ *is invariant under $\rho \mapsto 1-\rho$.*

(ii) $\operatorname{ord}_{1/2} G$ *is odd.*
(iii) No Weil datum of type $(1,0)$ *satisfies AW1 for* $U = \mathbb{1}$*, AW2 for* $(G, \mathbb{1})$*, and AW3.*

Proof of (i). The representation σ is self-dual (the remark after Theorem 7.3), so by Theorem 7.2(ii), $\operatorname{ord}_{1-\rho} \Lambda(\sigma, s) = \operatorname{ord}_{\rho} \Lambda(\sigma, s)$; and $\Lambda(s) = \Lambda(1-s)$ by Theorem 3.4. □

Proof of (ii). $\Lambda(\frac{1}{2}) \neq 0$ by Proposition 3.1. Since σ is self-dual with $W(\sigma) = -1$, the meromorphic function $t \mapsto A_\sigma^{t/2} \Lambda(\sigma, \frac{1}{2} + t)$ is odd (the consequence of Theorem 7.2(ii) recorded after it), and it is not identically zero because the Euler product does not vanish for $\operatorname{Re} s > 1$. An odd meromorphic function $ct^m + O(t^{m+1})$ with $c \neq 0$ has $(-1)^m = -1$, so $\operatorname{ord}_{1/2} \Lambda(\sigma, s)$ is odd, and so is $\operatorname{ord}_{1/2} G$. □

Proof of (iii). By AW2 with $d = 1$, $\operatorname{div} G = -\operatorname{Sp}_0 + \operatorname{Sp}_1 - \operatorname{Sp}_2$, and $\operatorname{Sp}_0 = [0]$, $\operatorname{Sp}_2 = [1]$ by AW1; so the multiplicity $m_1(\frac{1}{2})$ equals $\operatorname{ord}_{1/2} G$, which is odd by (ii). By graded commutativity in AW1, the pairing $B_1(u, v) = \operatorname{Tr}(u \cup v)$ on V^1 is alternating, and by AW3 it restricts to a perfect pairing of $V^1_{\Theta \sim 1/2}$ with itself. A perfect alternating form exists only in even dimension, a contradiction. □

Proposition 7.6(iii) therefore does not apply to G; the hypothesis "even central order" of Definition 7.5 is a root-number condition, the same parity that Proposition 7.15 below reads off the central space.

7.4. **Galois descent.** For a finite Galois extension L/K with group G, each $g \in G$ induces an automorphism of $\overline{X}_L$ over $\overline{X}_K$, so a contravariant assignment on $\overline{\mathrm{Sch}}_1$ makes G act on $V_L^\bullet$ (by pullback along g^{-1}), commuting with Θ and $\cup$. Every generalized eigenspace of Θ_1 on V_L^1 is then a finite-dimensional representation of G. The character values of a finite group are algebraic numbers.

Definition 7.10. Let $X \mapsto (H^\bullet(\overline{X}), \Theta, \cup, \mathrm{Tr}, *)$ be a contravariant assignment of Weil data of type $(1,0)$ on a full subcategory of $\overline{\mathrm{Sch}}_1$.

(a) The assignment satisfies *Galois descent* if, for every finite Galois extension L'/K' of fields whose objects lie in the subcategory, with group G', pullback along $\overline{X}_{L'} \to \overline{X}_{K'}$ induces an isomorphism $H^\bullet(\overline{X}_{K'}) \cong H^\bullet(\overline{X}_{L'})^{G'}$ compatible with Θ and $\cup$.

Let moreover L/K be a finite Galois extension of number fields with group G such that $\overline{X}_L$ lies in the subcategory.

(b) For an irreducible representation σ of G, the space $M_\sigma = \operatorname{Hom}_G(V_\sigma, H^1(\overline{X}_L))$, with Θ acting by composition, is the *isotypic multiplicity space* of σ.
(c) The *central space* is the generalized eigenspace $V_{1/2} = H^1(\overline{X}_L)_{\Theta \sim 1/2}$, a representation of G.
(d) The *rational orbit* $O(\sigma)$ of an irreducible representation σ of G is the set of irreducible representations τ of G whose character is $\gamma \circ \chi_\sigma$ for some field automorphism γ of $\mathbb{C}$, and $\chi_{O(\sigma)} = \sum_{\tau \in O(\sigma)} \chi_\tau$.

Each M_σ is a spectral core, and $H^1(\overline{X}_L) \cong \bigoplus_\sigma V_\sigma \otimes M_\sigma$ compatibly with Θ [Den23, p. 4]. The set $O(\sigma)$ is finite, and $\chi_{O(\sigma)}$ takes rational values; if χ_σ is rational-valued, $O(\sigma) = \{\sigma\}$. The functoriality expectations behind Definition 7.10 are [Den23, p. 4] and [Den25, §§2.8–2.10].

It is natural to expect that functoriality in the field, added to the axioms on each object, contributes transcendental content beyond zero location; in fact, without positivity, it contributes Artin's holomorphy conjecture for the rational-valued characters and the corresponding isotypic multiplicities, and no information on the location of zeros.

Proposition 7.11 (Artin; [Ser77, §13.1, Theorems 29–30, pp. 102–103])**.** *Let G be a finite group and σ an irreducible representation of G. Then $\chi_{O(\sigma)}$ is a $\mathbb{Q}$-linear combination of the permutation characters $\chi_{\mathrm{Ind}_H^G 1}$, H running over the cyclic subgroups of G.*

Proof. Call $g, g' \in G$ *cyclically equivalent* if $\langle g\rangle$ and $\langle g'\rangle$ are conjugate. The function $\chi_{O(\sigma)}$ is constant on cyclic equivalence classes: it is a class function, and if g has order n and k is prime to n, the eigenvalues of $\tau(g^k)$ are the k-th powers of those of $\tau(g)$, so $\chi_\tau(g^k) = \gamma_k(\chi_\tau(g))$ for an automorphism γ_k of $\mathbb{C}$ sending every n-th root of unity ζ to ζ^k; since γ_k permutes $O(\sigma)$, $\chi_{O(\sigma)}(g^k) = \chi_{O(\sigma)}(g)$. The permutation character $\chi_{\mathrm{Ind}_H^G 1}(g) = \#\{xH \in G/H : x^{-1}gx \in H\}$ is also constant on these classes, and is nonzero exactly when $\langle g\rangle$ is conjugate to a subgroup of H. Let $H_1, \dots, H_r$ represent the conjugacy classes of cyclic subgroups, ordered by nondecreasing order, and g_i a generator of H_i. The matrix $(\chi_{\mathrm{Ind}_{H_i}^G 1}(g_k))_{i,k}$ has integer entries, and an entry with $k > i$ vanishes, since then H_k is not conjugate to a subgroup of H_i; the diagonal entries are at least one. So the matrix is invertible over $\mathbb{Q}$, the r permutation characters span the r-dimensional space of rational-valued functions on the cyclic equivalence classes, and $\chi_{O(\sigma)}$ lies in this space. □

Proposition 7.12. *Let L/K be a finite Galois extension of number fields with group G, and suppose a contravariant assignment on the full subcategory of $\overline{\mathrm{Sch}}_1$ with objects $\overline{X}_{L^H}$, $H \subset G$, satisfies Galois descent and, on each $\overline{X}_{L^H}$, AW1 and AW2 for $(\Lambda_{L^H}, \nVdash)$, and let M_σ and $O(\sigma)$ be as in Definition 7.10. Then for every irreducible representation σ of G, with completed Artin L-functions $\Lambda(\tau, s)$ of L/K (Definition 7.1),*

$$\sum_{\tau \in O(\sigma)} \mathrm{Sp}(\Theta \mid M_\tau) = \mathrm{div} \prod_{\tau \in O(\sigma)} \Lambda(\tau, s) + \delta_{\sigma,1}\big([0] + [1]\big).$$

Consequently:

- *(i) if χ_σ is rational-valued, then for every ρ (with $\rho \notin \{0, 1\}$ if $\sigma = 1$) the algebraic multiplicity of ρ on M_σ is $\mathrm{ord}_\rho \Lambda(\sigma, s)$;*
- *(ii) for $\sigma \neq 1$ the product $\prod_{\tau \in O(\sigma)} \Lambda(\tau, s)$ is entire; in particular $\Lambda(\sigma, s)$ is entire whenever χ_σ is rational-valued and $\sigma \neq 1$.*

Proof. Let $\Gamma \subset G$. By descent for L/L^Γ and the decomposition of $H^1(\overline{X}_L)$,

$$H^1(\overline{X}_{L^\Gamma}) \cong H^1(\overline{X}_L)^\Gamma \cong \bigoplus_\sigma V_\sigma^\Gamma \otimes M_\sigma,$$

and $\dim V_\sigma^\Gamma = \langle \sigma, \mathrm{Ind}_\Gamma^G 1\rangle$ by Frobenius reciprocity, so

$$\mathrm{Sp}(\Theta_1 \mid H^1(\overline{X}_{L^\Gamma})) = \sum_\sigma \langle \sigma, \mathrm{Ind}_\Gamma^G 1\rangle \,\mathrm{Sp}(\Theta \mid M_\sigma).$$

The proof of Theorem 6.3(i), which uses only AW1 and AW2, gives $\mathrm{Sp}(\Theta_1 \mid H^1(\overline{X}_{L^\Gamma})) = \mathrm{div}\,\Lambda_{L^\Gamma} + [0] + [1]$. By (7.1) and $\langle 1, \mathrm{Ind}_\Gamma^G 1\rangle = 1$,

$$\mathrm{div}\,\Lambda_{L^\Gamma} + [0] + [1] = \sum_\sigma \langle \sigma, \mathrm{Ind}_\Gamma^G 1\rangle \big(\mathrm{div}\,\Lambda(\sigma, s) + \delta_{\sigma,1}([0] + [1])\big).$$

Put $\Delta(\tau) = \mathrm{Sp}(\Theta \mid M_\tau) - \mathrm{div}\,\Lambda(\tau, s) - \delta_{\tau,1}([0] + [1])$, a divisor, and for a virtual character χ put $\Phi(\chi) = \sum_\tau \langle \tau, \chi\rangle \Delta(\tau)$, a $\mathbb{Z}$-linear function. We have shown $\Phi(\chi_{\mathrm{Ind}_\Gamma^G 1}) = 0$ for every subgroup Γ. By Proposition 7.11 there is an integer $n \geq 1$ with $n\chi_{O(\sigma)}$ an integral combination of permutation characters, so $0 = \Phi(n\chi_{O(\sigma)}) = n \sum_{\tau \in O(\sigma)} \Delta(\tau)$, and $\sum_{\tau \in O(\sigma)} \Delta(\tau) = 0$ because the group of divisors is torsion-free. This is the displayed identity (note that $O(1) = \{1\}$). If χ_σ

is rational-valued, $O(\sigma) = \{\sigma\}$, which gives (i). For (ii), the left side of the displayed identity is effective. □

Proposition 7.12 corrects the expectation that descent forces every isotypic divisor: the argument controls only sums over rational orbits, since the objects $\overline{X}_{L^H}$ see only permutation characters, which are rational-valued. With descent, AW1–AW2 imply the holomorphy of these orbit products without any positivity; positivity is needed only for the location of zeros. The proposition also shows that AW2 cannot be weakened to an inclusion of the zeros in the spectrum without losing (ii). Conversely, at the level of one finite Galois extension of $\mathbb{Q}$, the Riemann hypothesis and Artin's conjecture for its Artin L-functions produce the package with descent.

Definition 7.13. For a finite Galois extension $L/\mathbb{Q}$, let $\mathcal{A}_L$ be the full subcategory of $\overline{\mathrm{Sch}}_1$ whose objects are the $\overline{X}_K$ with $K \subset L$.

Proposition 7.14 (compare [Den25, Theorem 2.15])**.** *Let $L/\mathbb{Q}$ be a finite Galois extension with group G, and let $\Lambda(\sigma, s)$ be the completed Artin L-functions over $\mathbb{Q}$ of the irreducible representations σ of G. Suppose that for every irreducible $\sigma \neq 1$ the function $\Lambda(\sigma, s)$ is entire and all its zeros have real part $\frac{1}{2}$, and that all zeros of ξ have real part $\frac{1}{2}$. Then there is a contravariant assignment on $\mathcal{A}_L$ of Weil data of type $(1, 0)$, satisfying Galois descent, whose value on each $\overline{X}_K$ satisfies AW1–AW4 for $(\Lambda_K, \nVdash)$, with pullbacks commuting with Θ and $\cup$.*

Proof. For irreducible σ put $\Sigma_\sigma = \operatorname{div}\Lambda(\sigma, s)$ if $\sigma \neq 1$ and $\Sigma_1 = \operatorname{div}\xi$; by hypothesis these are effective divisors supported on $\operatorname{Re} s = \frac{1}{2}$. Let M_σ be the algebraic direct sum of lines $\mathbb{C}e$, one for each point of Σ_σ counted with multiplicity, with $\Theta e = \rho e$ on the lines over ρ, and write $M_{\sigma,\rho}$ for the span of the lines over ρ. Put $H^1(\overline{X}_L) = \bigoplus_\sigma V_\sigma \otimes M_\sigma$ with G acting on the first factors and Θ on the second; $H^0(\overline{X}_L) = \mathbb{C} \cdot 1$ and $H^2(\overline{X}_L) = \mathbb{C}\eta$ with trivial action, $\Theta_0 = 0$, $\Theta_2 = 1$, $\operatorname{Tr}(c\eta) = c$. This is a spectral core, since only finitely many σ occur.

Positive form. Let h be the orthogonal sum of tensor products of G-invariant inner products on the V_σ with the standard forms on the M_σ. It is positive, G-invariant, and satisfies (1.1) with weight one, because all eigenvalues lie on $\operatorname{Re} s = \frac{1}{2}$ and distinct eigenspaces are orthogonal.

Alternating duality. By Theorem 7.2(ii) (and $\xi(s) = \xi(1-s)$ for $\sigma = 1$), $\dim M_{\sigma^\vee,1-\rho} = \dim M_{\sigma,\rho}$. The involution $(\sigma, \rho) \mapsto (\sigma^\vee, 1-\rho)$ acts on the set of summands $V_\sigma \otimes M_{\sigma,\rho}$. For a pair of distinct summands exchanged by it, choose an isomorphism $M_{\sigma^\vee,1-\rho} \cong M_{\sigma,\rho}^*$, let β be the resulting G-invariant perfect pairing of the two summands (built from the canonical pairing of V_σ with $V_{\sigma^\vee} \cong V_\sigma^*$), and put $B(x, y) = \beta(x, y) = -B(y, x)$ for x in the first summand and y in the second. A summand fixed by the involution has $\sigma \cong \sigma^\vee$ and $\rho = \frac{1}{2}$. By Theorem 7.3(i), V_σ carries a nondegenerate G-invariant bilinear form b_σ that is either alternating, or symmetric with σ defined over $\mathbb{R}$. If it is alternating, let c be a nondegenerate symmetric form on $M_{\sigma,1/2}$. If it is symmetric and $\sigma \neq 1$, then $W(\sigma) = 1$ by Theorem 7.3(ii), so $t \mapsto A_\sigma^{t/2}\Lambda(\sigma, \frac{1}{2} + t)$ is even (consequence of Theorem 7.2(ii)), hence $\dim M_{\sigma,1/2} = \operatorname{ord}_{1/2}\Lambda(\sigma, s)$ is even, and we let c be a nondegenerate alternating form on $M_{\sigma,1/2}$ (for $\sigma = 1$ use $\xi(s) = \xi(1-s)$ instead). Put $B = b_\sigma \otimes c$ on the summand, which is then alternating and G-invariant. Let B vanish on all other pairs of summands. Then B is alternating, G-invariant, supported on pairs of eigenvalues with sum one, and pairs $V^1_{\Theta\sim\rho}$ perfectly with $V^1_{\Theta\sim 1-\rho}$. Put $u \cup v = B(u, v)\eta$ on $H^1(\overline{X}_L)$ and the unit products otherwise.

Star. Since B is perfect on each pair of eigenspaces with eigenvalues ρ and $1-\rho = \bar\rho$, there is a unique antilinear map $*_1$ with $B(u, *_1 v) = h(u, v)$ for all u, v; it maps the ρ-eigenspace to the $\bar\rho$-eigenspace, so $\Theta_1 *_1 = *_1 \Theta_1$, it is bijective because h is nondegenerate, and it is G-equivariant because B and h are G-invariant. With $*_0(c) = \bar c\eta$ and $*_2(c\eta) = \bar c$, the datum on $\overline{X}_L$ satisfies AW1, AW3, and AW4, as in the proof of Proposition 7.6.

The functor. For $K \subset L$ put $V_K^\bullet = (V_L^\bullet)^{\mathrm{Gal}(L/K)}$ with the restricted structure. For an embedding $K \hookrightarrow K'$ of subfields of L, extend it to $g \in G$ and let the pullback be $v \mapsto gv$; it does not depend on the extension, and it maps $(V_L)^{\mathrm{Gal}(L/K)}$ into $(V_L)^{\mathrm{Gal}(L/K')}$ because $g\,\mathrm{Gal}(L/K)g^{-1} = \mathrm{Gal}(L/gK) \supset \mathrm{Gal}(L/K')$. This is contravariant, commutes with Θ and $\cup$, and satisfies descent because invariants of invariants are invariants. On each V_K^1 the form h stays positive, $*_1$ preserves invariants, and $B(u, *_1 u) = h(u,u) > 0$ shows that B stays perfect on each pair of eigenspaces, so AW1, AW3, and AW4 hold on $\overline{X}_K$. Finally, with $H = \mathrm{Gal}(L/K)$, $\mathrm{Sp}_1(V_K^1) = \sum_\sigma \langle \sigma, \mathrm{Ind}_H^G 1 \rangle \Sigma_\sigma = \operatorname{div} \Lambda_K + [0] + [1]$ by (7.1), which is (AW-div) for Λ_K as in the proof of Theorem 6.3(i). □

It is natural to expect that a cohomology with real coefficients, complexified, could satisfy the package with descent; in fact Galois descent turns Deninger's no-go theorem into an inconsistency.

Proposition 7.15. *Let $L/\mathbb{Q}$ be a finite Galois extension with group G and σ an irreducible representation of G with rational-valued character, which is self-dual, not defined over $\mathbb{R}$, and has $W(\sigma) = -1$. Suppose a contravariant assignment on the full subcategory $\mathcal{A}_L$ of $\overline{\mathrm{Sch}}_1$ (Definition 7.13) satisfies Galois descent and, on each object, AW1 and AW2. Then σ occurs with odd multiplicity in the central space $V_{1/2}$ of Definition 7.10; consequently $V_{1/2}$ is not the complexification of a real representation of G. Such L and σ exist.*

Proof. The multiplicity of σ in $V_{1/2}$ is the dimension of the generalized $\frac{1}{2}$-eigenspace of M_σ, which is $\mathrm{ord}_{1/2}\,\Lambda(\sigma, s)$ by Proposition 7.12(i) with $K = \mathbb{Q}$. Since σ is self-dual and $W(\sigma) = -1$, the function $t \mapsto A_\sigma^{t/2}\Lambda(\sigma, \frac{1}{2} + t)$ is odd and not identically zero, so its order at $t = 0$ is odd. If $V_{1/2}$ were the complexification of a real representation, σ would occur in it with even multiplicity by Theorem 7.3(iii). The quaternion extension of Theorem 7.3(iv) with its two-dimensional representation satisfies the hypotheses (see the remark after that theorem). □

This is the argument of Deninger's no-go theorem (Theorem 11.1), whose hypothesis 2.12 [Den23] is here supplied by descent for the one rational character that the argument needs. The obstruction needs the class to contain the quaternion fields; for the single object $\overline{X}_{\mathbb{Q}}$ it says nothing (compare Remark 9.21).

Proposition 7.16 (after [Den23, 2.7–2.8, pp. 3–4])**.** *Let L/K be a finite Galois extension of number fields with group G, and suppose that a contravariant assignment on the full subcategory of $\overline{\mathrm{Sch}}_1$ with objects $\overline{X}_{L^H}$, $H \subset G$, satisfies Galois descent and, on each object, AW1 and AW2, and that the star $*_1$ on V_L^1 commutes with G and with Θ_1 on the central space $V_{1/2}$ of Definition 7.10 and satisfies $*_1^2 = -1$ there. Let σ be an irreducible representation of G defined over $\mathbb{R}$. Then σ occurs in $V_{1/2}$ with even multiplicity; if moreover χ_σ is rational-valued, $\mathrm{ord}_{1/2}\,L(\sigma, s)$ is even.*

Proof. The space $V_{1/2}$ is G-stable and $*_1$-stable (an antilinear map commuting with Θ_1 maps the ρ-generalized eigenspace to the $\bar\rho$ one). Write $V_\sigma = V_0 \otimes_{\mathbb{R}} \mathbb{C}$ and let j_σ be the G-equivariant complex conjugation of this real form, so $j_\sigma^2 = 1$. The σ-isotypic part of $V_{1/2}$ is $V_\sigma \otimes N$ with N the generalized $\frac{1}{2}$-eigenspace of M_σ, and $*_1$ maps it to the $\bar\sigma$-isotypic part, which is the same part since $\bar\sigma \cong \sigma$. Let κ be complex conjugation on N in some basis. Then $(j_\sigma \otimes \kappa)^{-1} *_1$ is linear and G-equivariant on $V_\sigma \otimes N$, hence equals $1 \otimes A$ for a linear A by Schur's lemma, and $*_1 = j_\sigma \otimes j_N$ with $j_N = \kappa A$ antilinear. Then $-1 = *_1^2 = j_\sigma^2 \otimes j_N^2 = 1 \otimes j_N^2$, so $j_N^2 = -1$, a quaternionic structure on N, and $\dim N$ is even. If χ_σ is rational-valued, $\dim N = \mathrm{ord}_{1/2}\,\Lambda(\sigma, s) = \mathrm{ord}_{1/2}\,L(\sigma, s)$ by Proposition 7.12(i), the gamma factors being holomorphic and nonzero at $\frac{1}{2}$. □

For σ self-dual and not defined over $\mathbb{R}$ the same argument, with an antilinear j_σ of square -1, gives a real structure on N and no parity constraint. The prediction is consistent with the functional equation: representations defined over $\mathbb{R}$ have $W(\sigma) = 1$ (Theorem 7.3(ii)) and hence even central order, while the others may have $W(\sigma) = -1$ and odd central order (Proposition 7.15). Root numbers therefore enter the package through the parity of central orders, never through the location of zeros.

7.5. **The finite-field instance.** This subsection places Weil's theorem in the frame of Definition 1.4: for a smooth projective geometrically connected variety over $\mathbb{F}_q$, the logarithmic suspension of ℓ-adic cohomology is a Weil datum of type $(d, 0)$ which satisfies AW1–AW3 by Grothendieck's formalism, whatever its stars, and whose positivity axiom holds exactly when Frobenius is semisimple and pure. The determinant identity holds with normalization term 1. We first record the algebra needed to suspend cup products and stars.

Proposition 7.17. *Let X, ℓ, $H^\bullet$, F, and tr be as in Theorem 4.9, let $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ be a field isomorphism, and write H^j also for $H^j \otimes_{\mathbb{Q}_\ell,\iota} \mathbb{C}$, with the induced F, cup product, and trace.*

(i) The cup product maps $H^i_{(\alpha)} \times H^k_{(\beta)}$ into $H^{i+k}_{(\alpha\beta)}$.

(ii) On each $H^j_{(\alpha)}$ let $N = \log(\alpha^{-1}\mathrm{F})$ as in Definition 4.15. Then $N(u \cup v) = Nu \cup v + u \cup Nv$ for all u, v.

(iii) If h_j is a positive Frobenius polarization of weight j on (H^j, F), there is a unique antilinear map $*_j : H^j \to H^{2d-j}$ with $\mathrm{tr}(u \cup *_j v) = h_j(u, v)$ for all u, v, and it maps $H^j_{(q^\rho)}$ into $H^{2d-j}_{(q^{d-\rho})}$ for every ρ with q^ρ an eigenvalue.

Proof of (i). By Theorem 4.9, F is an automorphism of the graded algebra $H^\bullet$. If $(\mathrm{F} \otimes \mathrm{F} - \alpha\beta)^n(u \otimes v) = 0$, then $(\mathrm{F} - \alpha\beta)^n(u \cup v) = 0$; and $\mathrm{F} \otimes \mathrm{F}$ has the single eigenvalue $\alpha\beta$ on $H_{(\alpha)} \otimes H_{(\beta)}$. □

Proof of (ii). By (i) the semisimple part F_s, equal to α on $H_{(\alpha)}$, is multiplicative, hence so is the unipotent part $\mathrm{F}_u = \mathrm{F}_s^{-1}\mathrm{F}$, and $N = \log \mathrm{F}_u$ on each $H_{(\alpha)}$. For integers $t \geq 0$, $\exp(tN) = \mathrm{F}_u^t$ is multiplicative; both sides of $\exp(tN)(u \cup v) = \exp(tN)u \cup \exp(tN)v$ are polynomials in t because N is nilpotent, so the identity holds for all real t, and differentiating at $t = 0$ gives the claim. □

Proof of (iii). The Poincaré pairing is perfect, so $*_j$ exists, is unique, and is antilinear. By Proposition 4.2, F is semisimple on H^j with eigenvalues of modulus $q^{j/2}$. From $h_j(\mathrm{F}u, \mathrm{F}v) = q^j h_j(u, v)$ and $\mathrm{tr}(\mathrm{F}u \cup \mathrm{F}w) = q^d \,\mathrm{tr}(u \cup w)$ we get $\mathrm{tr}(\mathrm{F}u \cup *_j \mathrm{F}v) = q^{j-d}\,\mathrm{tr}(\mathrm{F}u \cup \mathrm{F} *_j v)$ for all u, so $*_j\mathrm{F} = q^{j-d}\mathrm{F}*_j$. For an eigenvector v with eigenvalue $\alpha = q^\rho$, $\mathrm{Re}\,\rho = j/2$, this gives $\mathrm{F}(*_j v) = q^{d-j}\bar{\alpha} \; *_j v = q^{d-\rho} \; *_j v$, because $q^{\bar{\rho}} = \bar{\alpha}$ and $\bar{\rho} = j - \rho$. □

Definition 7.18. Let X, ℓ, ι, $H^\bullet$, F, and tr be as in Proposition 7.17, and let $(\widehat{H}^j, \Theta_j)$ be the logarithmic suspension of (H^j, F) (Definition 4.15).

(a) The *suspended datum* of (X, ℓ, ι) is the Weil datum $(\widehat{H}^\bullet, \Theta, \cup, \mathrm{Tr}, 0)$ of type $(d, 0)$ with zero stars, where for x the copy of $u \in H^i_{(q^\rho)}$ in $\widehat{H}^i_\rho$ and y the copy of $v \in H^k_{(q^{\rho'})}$ in $\widehat{H}^k_{\rho'}$, the product $x \cup y$ is the copy of $u \cup v$ in $\widehat{H}^{i+k}_{\rho+\rho'}$ if $q^{\rho+\rho'}$ is an eigenvalue of F on H^{i+k} and 0 otherwise, and $\mathrm{Tr}(x) = \mathrm{tr}(u)$ if x is the copy of $u \in H^{2d}$ in the summand $\widehat{H}^{2d}_d$ and $\mathrm{Tr}(x) = 0$ on the other summands.

(b) If positive Frobenius polarizations h_j of weight j on (H^j, F) are given for all j, the *suspended stars* are the antilinear maps $\widehat{*}_j$ sending the copy of $v \in H^j_{(q^\rho)}$ in $\widehat{H}^j_\rho$ to the copy of $*_j v$ in $\widehat{H}^{2d-j}_{d-\rho}$, with $*_j$ as in Proposition 7.17(iii).

By Proposition 7.17, the product lands in the indicated summand, the suspended stars are well defined, and $\widehat{H}_d^{2d}$ is a summand because $\mathrm{F} = q^d$ on H^{2d}. Axioms AW1–AW3 and AW-det do not involve the stars. The suspended datum is Deninger's cohomology $H^\bullet(X/L) = D(H^\bullet(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) \otimes \mathbb{C})$, which inherits products and duality from ℓ-adic cohomology because D is an exact tensor functor [Den94, §4, pp. 16–17].

The regularized determinants of suspended Frobenius are computed first. Recall the Hurwitz zeta function $\zeta_H(w, a) = \sum_{n \geq 0} (n + a)^{-w}$ for $a \in \mathbb{C} \setminus (-\infty, 0]$ and $\operatorname{Re} w > 1$, with principal powers.

Theorem 7.19 (Lerch; [Den94, proof of (2.7), p. 9])**.** *For $a \in \mathbb{C} \setminus (-\infty, 0]$, $\zeta_H(w, a)$ continues meromorphically in w to $\mathbb{C}$ with a single simple pole at $w = 1$, and $\zeta_H(0, a) = \frac{1}{2} - a$ and $\exp(\partial_w \zeta_H(0, a)) = \Gamma(a)/\sqrt{2\pi}$.*

The source states these classical facts without proof.

It is natural to expect that the identity $\det_\infty((s - \Theta)/2\pi) = 1 - \alpha q^{-s}$ holds only up to an unspecified normalization and that the factor 2π must be tuned; in fact, for the principal cut, the constant is exactly 1, and passing to the reflected cut changes the determinant by a factor e^{a+bs} with nonzero b.

Theorem 7.20 (Deninger; Hesselholt)**.** *Let $q > 1$ be real, V a finite-dimensional complex vector space, $\mathrm{F} : V \to V$ invertible, and $(\widehat{V}, \Theta)$ its logarithmic suspension. Then $(\widehat{V}, \Theta)$ is zeta-regularizable for its principal cut (Definition 5.3), and*

$$\det_\infty\big((s - \Theta)/2\pi\big) = \det(1 - \mathrm{F} q^{-s} \mid V) = \prod_\alpha (1 - \alpha q^{-s})^{\dim V_{(\alpha)}}.$$

Moreover $Z(0; s) = 0$ for every s in the domain of the cut, so the same identity holds with 2π replaced by any $\delta > 0$.

Proof. The spectrum is the union of the progressions $\Sigma_q(\alpha)$, with multiplicity $\dim V_{(\alpha)}$ on each point of $\Sigma_q(\alpha)$ (Proposition 4.16(i)), and the spectral zeta function is the corresponding weighted sum over α; it suffices to treat one progression with multiplicity one. Fix a logarithm λ of α, put $\rho_n = (\lambda + 2\pi i n)/\log q$ for $n \in \mathbb{Z}$, and for $s \in \mathbb{C}$ put $\sigma = s \log q - \lambda$ and $z = i\sigma/2\pi$. Then

$$t_n = \frac{s - \rho_n}{2\pi} = \frac{-i}{\log q}(n + z), \qquad t_{-m} = \frac{i}{\log q}(m - z) \quad (m \geq 1).$$

Let $R = \{z' \in \mathbb{C} : \operatorname{Im} z' > 0\} \cup \{z' \in \mathbb{C} : 0 < \operatorname{Re} z' < 1\}$. Replacing λ by $\lambda + 2\pi i n_0$ replaces z by $z + n_0$ and only reindexes the spectrum. The point s lies in the domain U_{pr} of the principal cut exactly when no t_n lies in $(-\infty, 0]$, that is, unless $\operatorname{Re} z \in \mathbb{Z}$ and $\operatorname{Im} z \leq 0$; so for $s \in U_{\mathrm{pr}}$ there is n_0 with $z + n_0 \in R$, and such s form an open set. Assume $z \in R$. For $n \geq 0$ the principal argument of $n + z$ lies in $(-\pi/2, \pi]$, so the principal argument of t_n is $\arg(n + z) - \pi/2$ and $t_n^{-w} = (\log q)^w e^{i\pi w/2}(n + z)^{-w}$; for $m \geq 1$ the principal argument of $m - z$ lies in $(-\pi, \pi/2]$, so $t_{-m}^{-w} = (\log q)^w e^{-i\pi w/2}(m - z)^{-w}$. Moreover z and $1 - z$ lie outside $(-\infty, 0]$. Hence, for $\operatorname{Re} w > 1$,

$$Z(w; s) = \sum_{n \in \mathbb{Z}} t_n^{-w} = (\log q)^w \Big[e^{i\pi w/2} \zeta_H(w, z) + e^{-i\pi w/2} \zeta_H(w, 1 - z) \Big],$$

which by Theorem 7.19 continues meromorphically in w, is regular at $w = 0$, depends holomorphically on s near $w = 0$, and has $Z(0; s) = (\frac{1}{2} - z) + (\frac{1}{2} - (1 - z)) = 0$. Differentiating at $w = 0$, the factor $(\log q)^w$ contributes $\log \log q \cdot Z(0; s) = 0$, and

$$\partial_w Z(0; s) = \tfrac{i\pi}{2}\big[(\tfrac{1}{2} - z) - (z - \tfrac{1}{2})\big] + \partial_w \zeta_H(0, z) + \partial_w \zeta_H(0, 1 - z),$$

so, using $\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z$,

$$e^{-\partial_w Z(0;s)} = e^{-i\pi/2} e^{i\pi z} \frac{2\pi}{\Gamma(z)\Gamma(1-z)} = -ie^{i\pi z} \cdot 2\sin \pi z = 1 - e^{2\pi i z} = 1 - e^{-\sigma} = 1 - \alpha q^{-s}.$$

The right side does not depend on n_0, so $e^{-\partial_w Z(0;s)} = 1 - \alpha q^{-s}$ on all of U_{pr}, and it continues to that entire function. With δ in place of 2π, the factor $(\log q)^w$ becomes $(\delta \log q/2\pi)^w$, which again contributes 0 because $Z(0;s) = 0$. □

Theorem 7.20 is [Den94, (2.7), pp. 8–9, and (2.8), p. 10] and [Hes18, Proposition 7.1; p. 21 of the arXiv version]; the proof above follows the former.

Proposition 7.21 ([Den94, (2.10), p. 11, and (4.1), p. 16])**.** *Let $q > 1$, $\alpha \in \mathbb{C}^\times$, and let $(\widehat{V}, \Theta)$ be the logarithmic suspension of $(\mathbb{C}, \alpha)$. Then $(\widehat{V}, \Theta)$ is zeta-regularizable for its reflected cut (Definition 5.3), and*

$$\det\nolimits_\infty\big((s-\Theta)/2\pi\big) = 1 - \alpha^{-1} q^s = -\alpha^{-1} q^s\,(1 - \alpha q^{-s}).$$

Proof. For $w \notin [0,\infty)$ the argument of w in $[0, 2\pi)$ is $\mathrm{Arg}(-w) + \pi$. Hence, for $s \in U_{\mathrm{rf}}$, $((s-\rho)/2\pi)^{-w} = e^{-i\pi w}((\rho - s)/2\pi)^{-w}$ with the principal power on the right. The points $-\rho$, $\rho \in \Sigma_q(\alpha)$, form $\Sigma_q(\alpha^{-1})$, and $(\rho - s)/2\pi = ((-s) - (-\rho))/2\pi$, so for $s \in U_{\mathrm{rf}}$ the spectral zeta function of the reflected cut is $Z'(w;s) = e^{-i\pi w} Z^-(w;-s)$, where Z^- is the principal spectral zeta function of the suspension of $(\mathbb{C}, \alpha^{-1})$, and $-s$ lies in its principal domain. By Theorem 7.20, $Z^-(0;-s) = 0$ and $e^{-\partial_w Z^-(0;-s)} = 1 - \alpha^{-1} q^s$, so $-\partial_w Z'(0;s) = i\pi Z^-(0;-s) - \partial_w Z^-(0;-s)$ gives the claim. □

The two answers have the same divisor and differ by the factor $-\alpha^{-1} q^s = e^{a+bs}$ with $b = \log q \neq 0$: the normalization term of Definition 5.6 is a genuine convention and is recorded as data.

It is natural to expect that the suspension can be done with a single logarithm of each eigenvalue; in fact a single branch destroys the divisor.

Proposition 7.22. *Let $q > 1$, $\alpha \in \mathbb{C}^\times$, and $\rho_0 \in \Sigma_q(\alpha)$, and let $(V, \Theta) = (\mathbb{C}, \rho_0)$. Then (V, Θ) is a spectral core whose spectral divisor $[\rho_0]$ differs from the divisor $\sum_{\rho \in \Sigma_q(\alpha)} [\rho]$ of $1 - \alpha q^{-s}$, and $\det_\infty((s-\Theta)/2\pi) = (s - \rho_0)/2\pi$ for every spectral cut, a polynomial in s and not a function of q^{-s}.*

Proof. $\Sigma_q(\alpha)$ is infinite, and the determinant of a one-dimensional core was computed after Definition 5.2. □

The result also depends on the choice of ρ_0, which is defined only modulo $2\pi i/\log q$; the whole progression is needed, and with it Theorem 7.20 recovers the Euler factor exactly.

It is natural to expect that the finite-field zeta function satisfies the axioms verbatim, with $V^0 = \mathbb{C}$ and $V^{2d} = \mathbb{C}(-d)$; in fact the unit object must be the cohomology of the base, which over $\mathbb{F}_q$ is periodic.

Proposition 7.23 (Deninger [Den98, pp. 167–168 and Proposition 3.1])**.** *Let X be a smooth projective geometrically connected variety over $\mathbb{F}_q$, $\ell \nmid q$ a prime, $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ a field isomorphism, and $\widehat{H}^0$ the degree-zero part of the suspended datum (Definition 7.18). Then $\widehat{H}^0$, with its product, is isomorphic to the q-periodic unit $\mathbb{1}_q$ of Definition 1.3; its spectrum is $(2\pi i/\log q)\mathbb{Z}$; its principal-cut determinant is $1 - q^{-s}$, not $s/2\pi$; and it is not isomorphic to the arithmetic unit $\mathbb{1}$.*

Proof. By Theorem 4.9, $H^0 = \mathbb{C}\cdot 1$ with $\mathrm{F} = 1$, so $\widehat{H}^0$ has one line $\mathbb{C}1_\rho$ for each $\rho \in \Sigma_q(1) = (2\pi i/\log q)\mathbb{Z}$, with $\Theta = \rho$ there, and by Definition 7.18 $1_\rho \cup 1_{\rho'} = 1_{\rho+\rho'}$. Sending $1_{2\pi i n/\log q}$ to v^n is the isomorphism. The determinant is Theorem 7.20 with $\alpha = 1$. The spectrum of $\mathbb{1}$ is $\{0\}$. □

Theorem 7.24 (Theorem B; (i)–(iii) Deninger [Den94, §4, Proposition (4.1), and pp. 16–17])**.** *Let X be a smooth projective geometrically connected variety of dimension d over $\mathbb{F}_q$, $\ell \nmid q$ a prime, $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ a field isomorphism, H^j short for $H^j(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell,\iota} \mathbb{C}$ with geometric Frobenius* F*, and $(\widehat{H}^\bullet, \Theta, \cup, \mathrm{Tr}, 0)$ the suspended datum (Definition 7.18). Then:*

- *(i) it satisfies AW1 for $(\zeta_X, \mathbb{1}_q)$;*
- *(ii) it satisfies AW-det for ζ_X with the principal cuts and normalization term 1, with $D_j(s) = \det(1 - \mathrm{F}q^{-s} \mid H^j)$, and hence AW2;*
- *(iii) it satisfies AW3 with $w = 0$;*
- *(iv) for each j, the following are equivalent: the datum satisfies AW4-V in degree j; it satisfies AW4-E in degree j;* F *is semisimple on H^j and $|\alpha| = q^{j/2}$ for every eigenvalue α. If they hold for every j, the datum with the suspended stars of Definition 7.18 satisfies AW4;*
- *(v) if the datum satisfies AW4-V in degree j, every eigenvalue α of* F *on H^j satisfies $|\alpha| = q^{j/2}$;*
- *(vi) if X is a curve, positive Frobenius polarizations exist in every degree, and the datum with the suspended stars satisfies AW1–AW4.*

Proof of (i). By Proposition 7.17(i) and Theorem 4.9, the product is associative and graded-commutative with unit the copy of 1 in $\widehat{H}^0_0$. Since $\Theta = \rho + N/\log q$ on $\widehat{H}_\rho$, indices add under the product, and N is a derivation (Proposition 7.17(ii)), Θ is a derivation. By Theorem 4.9, $H^0 = \mathbb{C}$ with $\mathrm{F} = 1$, so $\widehat{H}^0 \cong \mathbb{1}_q$ by Proposition 7.23; and H^{2d} is a line with $\mathrm{F} = q^d$, so $\widehat{H}^{2d}$ has one line over each point of $\Sigma_q(q^d) = d + (2\pi i/\log q)\mathbb{Z}$, which is $\mathbb{1}_q(-d)$. □

Proof of (ii). By Theorem 7.20, $D_j(s) = \det(1 - \mathrm{F}q^{-s} \mid H^j)$, whose divisor is Sp_j by Proposition 4.16(i). By Theorem 3.8, $\zeta_X(s) = \prod_j \det(1 - \mathrm{F}q^{-s} \mid H^j)^{(-1)^{j+1}}$, which is (AW-det) with $a = b = 0$. □

Proof of (iii). Tr vanishes off $\widehat{H}^{2d}_d$, where $\Theta = d$, so $\mathrm{Tr}(\Theta x) = d\,\mathrm{Tr}(x)$. For $x \in \widehat{H}^j_\rho$ and $y \in \widehat{H}^{2d-j}_{\rho'}$, $x \cup y$ lies in the summand $\rho + \rho'$, so $\mathrm{Tr}(x \cup y)$ vanishes unless $\rho' = d - \rho$, and then equals $\mathrm{tr}(u \cup v)$. The Poincaré pairing satisfies $\mathrm{tr}(\mathrm{F}u \cup \mathrm{F}v) = q^d\,\mathrm{tr}(u \cup v)$, so it vanishes on the image of $\mathrm{F} \otimes \mathrm{F} - q^d$, which contains $H^j_{(\alpha)} \otimes H^{2d-j}_{(\beta)}$ when $\alpha\beta \neq q^d$. Since the pairing is perfect (Theorem 4.9), its restriction to $H^j_{(q^\rho)} \times H^{2d-j}_{(q^{d-\rho})}$ is perfect, which by Proposition 4.16(ii) is the pairing of $\widehat{H}^j_{\Theta\sim\rho}$ with $\widehat{H}^{2d-j}_{\Theta\sim d-\rho}$. □

Proof of (iv) and (v). If AW4-V holds in degree j, Proposition 4.17 gives a positive Frobenius polarization of weight j on (H^j, F), so F is semisimple and pure by Proposition 4.2; this also proves (v). If F is semisimple and pure, Proposition 4.3 gives a positive Frobenius polarization and Proposition 4.16(iii) gives AW4-V. AW4-V and AW4-E are equivalent by Proposition 6.2. Suppose now that positive Frobenius polarizations h_j exist for all j, and let $*_j$ be as in Proposition 7.17(iii), so that $\widehat{*}_j$ maps $\widehat{H}^j_\rho$ to $\widehat{H}^{2d-j}_{d-\rho}$ and every eigenvalue ρ has $\mathrm{Re}\,\rho = j/2$. On $\widehat{H}^j_\rho$ we have $\Theta_{2d-j}\widehat{*}_j = (d-\rho)\widehat{*}_j$ and $\widehat{*}_j\Theta_j = \bar{\rho}\,\widehat{*}_j$, and $d - \rho = \bar{\rho} + (d-j)$: this is the star law. Finally, for x the copy of u over ρ and y the copy of v over ρ', $x \cup \widehat{*}_j y$ lies over $\rho + d - \rho'$, so $\mathrm{Tr}(x \cup \widehat{*}_j y)$ is $h_j(u, v)$ if $\rho = \rho'$ and 0 otherwise: the star form is the suspended form $\widehat{h}_j$, which is positive definite. □

Proof of (vi). For $j = 0, 2$, F is 1 and q on a line. For $j = 1$, if C has genus 0 then $H^1 = 0$ by the identification recalled after Proposition 4.20. Otherwise, by Proposition 4.20, $(V_\ell J \otimes_\iota \mathbb{C}, \pi)$ satisfies AW4-$\mathrm{E_F}$ of weight one; under the identification of H^1 with the dual of $V_\ell J \otimes_\iota \mathbb{C}$, with F the transpose of π (the imported identification recalled after Proposition 4.20), Proposition 4.19(iv) shows that (H^1, F) satisfies AW4-$\mathrm{E_F}$ of weight one. By Proposition 4.19(i), F is semisimple on H^1 with eigenvalues of modulus $q^{1/2}$, and by Proposition 4.19(iii) the datum satisfies AW4-E in degree one directly. Positive Frobenius polarizations then exist in every degree by Proposition 4.3; apply (iv) and (i)–(iii). □

Thus for curves the suspended datum satisfies all four axioms, and the finite-field Riemann hypothesis is the conclusion of the purity argument applied to it. Weil's positivity of the Rosati involution is an instance of the multiplicative algebra form AW4-$\mathrm{E_F}$ (Proposition 4.20); the additive algebra form AW4-E on the suspension follows from it inside the same algebra (Proposition 4.19), and the star form AW4 from AW4-E through (iv), with the non-canonical polarizations of Proposition 4.3. The only import in this chain is the Frobenius compatibility of the identification of H^1 with the dual of the Tate module. For a general X the statement is weaker: AW1–AW3 are theorems, the conclusion $|\alpha| = q^{j/2}$ is Deligne's theorem, but the hypothesis AW4 in degree j is *equivalent* to Deligne's theorem together with the semisimplicity of Frobenius on H^j, which is open in general (Remark 4.13).

It is natural to expect that, since Deligne proved $|\alpha| = q^{j/2}$, the finite-field instance of the positivity axiom is a theorem for every smooth projective variety; in fact it is known only where Frobenius is known to be semisimple.

Remark 7.25. By Theorem 7.24(iv) and Remark 4.13, the status of the axioms for a smooth projective $X/\mathbb{F}_q$ is: AW1–AW3 and AW-det hold for all X; AW4 in degree j holds if and only if Frobenius is semisimple on H^j (purity being Deligne's theorem), which is known for curves and abelian varieties and implied by the standard conjectures [Kle94, p. 10], [Mil15, Aside 3.18, p. 42]. Deligne's proof of purity does not pass through a positive form. What Weil proved for curves, and what the standard conjectures would give in general, is positivity of an involution on an algebra of correspondences (Proposition 4.8), never a positive form on H^j itself: the Weil form of the standard conjectures is positive definite on some summands of H^j and negative definite on others [Mil15, Definition 3.6, p. 39, Theorem 3.15, and fn. 36, p. 42]. The arithmetic package therefore mirrors the conjectural route of Serre, Grothendieck, and Kleiman, not the proved route of Deligne.

Remark 7.26. Positivity is imposed on the complex side, for one field isomorphism ι at a time, and gives $|\iota(\alpha)| = q^{j/2}$ for that ι. Weil's Riemann hypothesis asks this for every complex conjugate of every eigenvalue; it follows from the statement for one ι once the characteristic polynomial is known to have rational coefficients (Theorem 3.8), and the Rosati argument proves it for every embedding at once (Proposition 4.11). Compared with the finite-field formalism, the arithmetic axioms lose rationality and integrality in q^{-s}, finite dimensionality, and Frobenius as an operator (only the flow $e^{t\Theta}$ remains, with $\mathrm{F} = e^{\Theta \log q}$), and they gain the archimedean places, whose gamma factors, together with ζ_K, produce the poles of Λ_K at 0 and 1 that the endpoint blocks account for in the proof of Theorem 6.3. The integer-twist blindness of Proposition 9.3 is visible on the suspension: $\ker(\Theta - n)$ on $\widehat{H}^{2n}$ is the $\mathrm{F} = q^n$ eigenspace, so cones of integer twists never see $\operatorname{Im}\rho \neq 0$, although the determinants of Theorem 7.20 contain the whole spectrum.

7.6. **Higher-dimensional arithmetic schemes.** For a regular projective flat arithmetic scheme X of dimension d the Hasse–Weil zeta function $\zeta_X(s) = \prod_x (1 - N(x)^{-s})^{-1}$, the product over the closed points x with residue field of order $N(x)$, converges for $\operatorname{Re} s > d$; its completion $\widehat{\zeta}_X$ by Serre's gamma factors, its meromorphic continuation, and its functional equation are conjectural

in general [Den10, Conjecture 1, p. 2]. A Weil datum with AW2 for such an object presupposes the continuation, just as the existence of a Weil cohomology over $\mathbb{F}_q$ yields rationality.

It is natural to expect that all nontrivial zeros of the zeta function of a higher-dimensional arithmetic scheme lie on one line; in fact already for $\mathbb{P}^1_{\mathbb{Z}}$ they lie in two disjoint strips.

Proposition 7.27 (compare [Den10, Conjecture 1(c), p. 2])**.** *Let $X = \mathbb{P}^1_{\mathbb{Z}}$, so $d = 2$. Then $\zeta_X(s) = \zeta(s)\zeta(s-1)$. The zeros of ζ_X in $0 < \operatorname{Re} s < 2$ are exactly the points of $Z \cup (Z+1)$, where Z is the set of nontrivial zeros of ζ; the set Z is nonempty, Z lies in $0 < \operatorname{Re} s < 1$, and $Z+1$ in $1 < \operatorname{Re} s < 2$. In particular no vertical line contains all zeros of ζ_X in $0 < \operatorname{Re} s < 2$.*

Proof. The closed points of $\mathbb{P}^1_{\mathbb{Z}}$ lie over primes p, and grouping them by p gives $\zeta_X(s) = \prod_p Z(\mathbb{P}^1_{\mathbb{F}_p}, p^{-s})$ for $\operatorname{Re} s > 2$. Since $\#\mathbb{P}^1(\mathbb{F}_{p^n}) = p^n + 1$, $Z(\mathbb{P}^1_{\mathbb{F}_p}, t) = 1/((1-t)(1-pt))$, so $\zeta_X(s) = \prod_p (1-p^{-s})^{-1}(1-p^{1-s})^{-1} = \zeta(s)\zeta(s-1)$. In $0 < \operatorname{Re} s < 2$ the zeros of $\zeta(s)$ are the points of Z (in $0 < \operatorname{Re} s < 1$) together with no others, since ζ has no zeros on $\operatorname{Re} s \geq 1$ (Euler product for $\operatorname{Re} s > 1$; Theorem 3.4 for $\operatorname{Re} s = 1$); the zeros of $\zeta(s-1)$ there are the points $\rho + 1$ with ρ a zero of ζ in $-1 < \operatorname{Re} \rho < 1$, and ζ has no zeros with $-1 < \operatorname{Re} \rho \leq 0$ by the functional equation $\xi(s) = \xi(1-s)$ and the preceding sentence (the trivial zeros are at $-2, -4, \dots$, and at $\rho = 0$ one uses $\xi(0) \neq 0$, Proposition 3.6). So the zeros are $Z \cup (Z+1)$. Finally $Z \neq \emptyset$: otherwise ξ, an entire function of order one (Theorem 3.4), would have no zeros and would be e^{a+bs} by Hadamard's theorem [Dup17, Theorem 1.4, p. 2]; then $\xi(s) = \xi(1-s)$ forces $b = 0$, so ξ would be constant, whereas $\xi(s) \to \infty$ as $s \to +\infty$ along $\mathbb{R}$, because $\zeta(s) \geq 1$ for real $s > 1$ and $\pi^{-s/2}\Gamma(s/2) \to \infty$. □

Theorem 7.28 ((i) after Deninger [Den98, p. 172], [Den10, p. 5])**.** *Let $d \geq 1$, let $Z \not\equiv 0$ be meromorphic on $\mathbb{C}$, and let a Weil datum of type $(d, 0)$ satisfy the derivation rule of AW1, AW2 for Z, the trace equivariance of AW3, and AW4. Then:*

- *(i) every eigenvalue ρ of Θ_j has $\operatorname{Re} \rho = j/2$, and each Θ_j is diagonalizable;*
- *(ii) for every ρ with $j = 2\operatorname{Re}\rho \in \{0, 1, \dots, 2d\}$, $\operatorname{ord}_\rho Z = (-1)^{j+1} m_j(\rho)$, and Z has no zero or pole ρ with $2\operatorname{Re}\rho \notin \{0, 1, \dots, 2d\}$;*
- *(iii) hence every zero of Z has real part in $\{\frac{1}{2}, \frac{3}{2}, \dots, \frac{2d-1}{2}\}$, every pole has real part in $\{0, 1, \dots, d\}$, the zeros are exactly the odd-degree spectra and the poles exactly the even-degree spectra, and Z determines every spectral divisor Sp_j.*

Proof. By Proposition 6.1 with $w = 0$, h_j is a positive infinitesimal polarization of weight j, and Proposition 3.3 gives (i). By (i), the supports of the Sp_j lie on the pairwise disjoint lines $\operatorname{Re} s = j/2$, so in (AW-div) the coefficient of $[\rho]$ is the single term with $j = 2\operatorname{Re}\rho$, which gives (ii); and (iii) follows, the sign $(-1)^{j+1}$ being $+1$ for odd j and -1 for even j. □

Poles are part of the statement: for an arithmetic surface whose generic fibre is an elliptic curve E over $\mathbb{Q}$, $\zeta_X(s) = \zeta(s)\zeta(s-1)/L(E,s)$ up to finitely many Euler factors, and the zeros of $L(E,s)$ on $\operatorname{Re} s = 1$ are poles of ζ_X coming from V^2. The theorem uses no regularized determinant. Without positivity its last clause fails.

It is natural to expect that the determinant identity for the total zeta function of a higher-dimensional object determines the spectra in each degree, and that zeros and poles of different degrees may cancel in the alternating product regardless of positivity; in fact both depend on AW4: without it the degreewise spectra are not determined, and with it no cancellation occurs (Theorem 7.28).

Definition 7.29. Let $(V^\bullet, \Theta, \cup, \operatorname{Tr}, *)$ be a Weil datum of type $(2, 0)$ with unit $1 \in V^0 = \mathbb{C} \cdot 1$ for the cup product and $\Theta_0 = 0$, let $\eta \in V^4$ satisfy $\Theta_4 \eta = 2\eta$ and $\operatorname{Tr} \eta = 1$, and let $\rho \in \mathbb{C}$. The *adjunction of the pair* $\{\rho, 2-\rho\}$ is the tuple obtained by adding to V^1 and V^3 lines $\mathbb{C}e$ and $\mathbb{C}f$,

and to V^2 lines $\mathbb{C}g$ and $\mathbb{C}g'$, with $\Theta e = \rho e$, $\Theta f = (2-\rho)f$, $\Theta g = \rho g$, $\Theta g' = (2-\rho)g'$; with products $1 \cup x = x \cup 1 = x$ for the new vectors x, $e \cup f = \eta = -f \cup e$, $g \cup g' = \eta = g' \cup g$, and all other products involving a new vector equal to zero; and with the stars extended by zero on the new lines.

Proposition 7.30. *Let $Z \not\equiv 0$ be meromorphic on $\mathbb{C}$, let a Weil datum $(V^\bullet, \Theta, \cup, \mathrm{Tr}, *)$ of type $(2,0)$ satisfy AW1 for $U = \mathbb{1}\!\!\not\!\!\perp$, AW2 for Z, and AW3, and let $\rho \in \mathbb{C}$. Then there is $\eta \in V^4$ with $\Theta_4 \eta = 2\eta$ and $\mathrm{Tr}\,\eta = 1$, and the adjunction of the pair $\{\rho, 2-\rho\}$ (Definition 7.29) satisfies AW1 for $U = \mathbb{1}\!\!\not\!\!\perp$, AW2 for the same Z, and AW3, while its spectral divisors Sp_1, Sp_2, and Sp_3 differ from those of the given datum.*

Proof. By AW1, $V^4 \cong \mathbb{C}(-2)$ is a line with $\Theta_4 = 2$, and by AW3 the pairing $V^0 \times V^4 \to \mathbb{C}$, $(c, x) \mapsto \mathrm{Tr}(c \cup x)$, is perfect, so Tr is nonzero on V^4 and η exists. Every product of two new vectors that is not declared zero lands in V^4, and every product of three elements of positive degree involving a new vector vanishes (it lies in degree above 4 or involves a vanishing product), so the product is associative; the signs are those required by graded commutativity in degrees $1+3$ and $2+2$. The derivation rule holds on the new products because the eigenvalues of the factors add up to 2, and on products with the unit trivially. The endpoint conditions are unchanged. For AW3 the new lines pair perfectly, with eigenvalues $\rho \leftrightarrow 2-\rho$, in degrees $1 \leftrightarrow 3$ and $2 \leftrightarrow 2$, and trivially with the old ones. For AW2, the right side of (AW-div) changes by $[\rho] - ([\rho] + [2-\rho]) + [2-\rho] = 0$. The new lines raise multiplicities in degrees 1, 2, and 3, so Sp_1, Sp_2, and Sp_3 change. □

With AW4 the adjunction is impossible unless $\mathrm{Re}\,\rho = 1$ and $\mathrm{Re}(2-\rho) = 1$, in which case the new lines in V^1 and V^3 would violate Theorem 7.28(i). The same phenomenon for trace formulas: a Lefschetz or explicit-formula identity for X determines only the alternating sum $\sum_j (-1)^j m_j(\rho)$, which for $d = 1$ is the degree-one divisor, and for $d \geq 2$ becomes a degreewise statement only in combination with AW4.

Remark 7.31. Let M be a specified pure coefficient of weight w over a number field, with a specified meromorphic completed $L(M, s)$. If this function admits a polarized graded spectral realization of type $(1, w)$ in Definition 1.8, Theorem 1.15 gives $\mathrm{Re}\,\rho = (w+\nu)/2$ in degree ν. Thus its zeros lie on $\mathrm{Re}\,s = (w+1)/2$, and its poles can lie only on $\mathrm{Re}\,s = w/2$ and $\mathrm{Re}\,s = w/2 + 1$. This is the coefficient weight calculation of [Den94, (7.11)–(7.13), pp. 32–33] and [Den10, pp. 6–7]. It uses a spectral coefficient realization, not an internal unital Weil datum of positive weight. In Deninger's coefficient formalism the duality and star maps have dual or complementary coefficient targets. Such a coefficient realization is additional to the unweighted conjectures on $\overline{\mathrm{Sch}}^{\mathrm{proj}}$. The trivial coefficient has $w = 0$ and recovers the number-field case. An arithmetic Abel–Jacobi comparison is a further hypothesis in the conditional BSD rank argument.

7.7. **The normalization of the determinant.** Theorem 7.4 uses AW2 in the divisor form, so it is independent of any regularization. For AW-det on $\overline{X}_K$ the endpoint blocks give $D_0 = s/2\pi$ and $D_2 = (s-1)/2\pi$, and (AW-det) reads $D_1(s) = e^{a+bs}\xi_K(s)/4\pi^2$. Over $\mathbb{F}_q$ the normalization term is exactly 1 for the principal cut (Theorem 7.24); over a number field it is determined by the spectrum alone and is computed in Theorem 7.37.

Theorem 7.32 (Deninger; Soulé; compare Kurokawa; [Den92, Theorem 3.3, (3.3.1), p. 147], [Den98, pp. 164, 167, and p. 168])**.** *Let ρ run over the nontrivial zeros of ζ, with multiplicity. The zeta-regularized product $\prod_\rho (s-\rho)/2\pi$, formed with the arguments in $(-\pi, \pi]$, equals $\frac{s}{2\pi}\frac{s-1}{2\pi}\, 2^{-1/2}\pi^{-s/2}\Gamma(s/2)\zeta(s)$.*

The first source proves this formula for $\operatorname{Re} s > 1$ [Den92, Theorem 3.3, p. 147], and reports its extension to all of U_{pr} (the source's set Ω) in a letter of Soulé [Den92, Remark 3.4(1), p. 147]; for $\operatorname{Re} s > 1$ the first source takes the arguments in $(-\pi/2, \pi/2)$, which agrees there with the principal branch, and the convention $(-\pi, \pi]$ on U_{pr} is that of [Den98, p. 167]; the second restates it, attributing it to Deninger and to Schröter and Soulé [SS94], and notes that a related formula had been observed earlier by Kurokawa [Kur88, p. 24], for every number field, up to a constant and assuming the Riemann hypothesis. With ξ as in Definition 1.16, the right side is $2^{-1/2}\xi(s)/2\pi^2$. Theorem 7.37 below reproves it and extends it to every number field.

It is natural to expect that for $K = \mathbb{Q}$ the determinant of the nontrivial zeros is exactly $\xi/2\pi^2$, that is, that AW-det holds with normalization term 1 relative to Λ; in fact, by Theorem 7.32, the normalization term relative to Λ is the constant $2^{-1/2}$, and it is 1 relative to Deninger's completion (Theorem 7.37). The zero-location theorems use neither value.

The rest of this subsection computes the normalization term for every number field. It is stated with the gamma factors that zeta regularization singles out.

Definition 7.33. Let K be a number field with r_1 real and r_2 complex places, discriminant D_K, and degree $n = [K : \mathbb{Q}] = r_1 + 2r_2$.

(a) *Deninger's archimedean factors* are $\Gamma_{\mathbb{R}}^{\mathrm{D}}(s) = 2^{-1/2}\pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}^{\mathrm{D}}(s) = (2\pi)^{-s}\Gamma(s)$.

(b) *Deninger's completion* of ζ_K is $\widehat{\zeta}_K(s) = \Gamma_{\mathbb{R}}^{\mathrm{D}}(s)^{r_1}\Gamma_{\mathbb{C}}^{\mathrm{D}}(s)^{r_2}\zeta_K(s)$.

(c) Let (V, Θ) be a spectral core with spectrum $\operatorname{div}\xi_K$, counted with multiplicity $m(\rho) = \operatorname{ord}_\rho \xi_K$, and let $s \in \mathbb{C}$ be such that $\operatorname{Im} s \neq \operatorname{Im}\rho$ for every zero ρ of ξ_K. The *half-plane pieces* of its spectral zeta function for the principal cut are

$$Z_+(w; s) = \sum_{\operatorname{Im}\rho > \operatorname{Im} s} m(\rho)\Big(\frac{s-\rho}{2\pi}\Big)^{-w}, \qquad Z_-(w; s) = \sum_{\operatorname{Im}\rho < \operatorname{Im} s} m(\rho)\Big(\frac{s-\rho}{2\pi}\Big)^{-w},$$

with principal powers, wherever the series converge absolutely; so Z_+ collects the zeros above s.

(d) The *Stirling constants* of K are

$$A = \tfrac{n}{2}, \quad B = \tfrac{1}{2}\log|D_K| - \tfrac{n}{2}(1 + \log 2\pi), \quad C = 2 - \tfrac{r_1+r_2}{2}, \quad E = \tfrac{r_1}{2}\log 4\pi + \tfrac{r_2}{2}\log 8\pi.$$

(e) The *standard branch* of $\log\xi_K$ on $\operatorname{Re} z > 1$ is

$$f(z) = \log z + \log(z-1) + \tfrac{z}{2}\log|D_K| + r_1\big(-\tfrac{z}{2}\log\pi + \log\Gamma(\tfrac{z}{2})\big) + r_2\big(\log 2 - z\log 2\pi + \log\Gamma(z)\big) + \sum_{\mathfrak{p}}\sum_{k\geq 1}\frac{N\mathfrak{p}^{-kz}}{k},$$

with principal logarithms, $\log\Gamma$ the branch on $\mathbb{C}\setminus(-\infty, 0]$ that is real on $(0, \infty)$, and the sum over the nonzero prime ideals.

Deninger's factors are those of [Den91, p. 247] and Flach–Morin [FM20, (1), p. 521]; they are $\Gamma_{\mathbb{R}}^{\mathrm{D}} = 2^{-1/2}\Gamma_{\mathbb{R}}$ and $\Gamma_{\mathbb{C}}^{\mathrm{D}} = \frac{1}{2}\Gamma_{\mathbb{C}}$, so

$$\widehat{\zeta}_K(s) = 2^{-n/2}|D_K|^{-s/2}\Lambda_K(s). \tag{7.2}$$

The set of s in Definition 7.33(c) is $U_{\mathrm{pr}} \cap U_{\mathrm{rf}}$ for the spectrum $\operatorname{div}\xi_K$ (Definition 5.3).

The proofs use three classical inputs.

Theorem 7.34 (Stirling; Hadamard; [Hin10, Lemma 3.1, p. 24], [HSW21, p. 3], [Dup17, Theorem 1.4, p. 2, specialized to order one, and Lemma 4.10, p. 11])**.** *Let K be a number field.*

(i) With $\log\Gamma$ as in Definition 7.33(e), $\log\Gamma(z) = (z - \frac{1}{2})\log z - z + \frac{1}{2}\log 2\pi + R(z)$ with $|R(z)| \leq c_\delta/|z|$ for $|z| \geq 1$ and $|\arg z| \leq \pi - \delta$, for each $\delta > 0$.

(ii) ξ_K is an entire function of order one.

(iii) An entire function of order one with zeros $\rho \neq 0$ counted with multiplicity satisfies $\sum_\rho |\rho|^{-2} < \infty$ and is $e^{a+bs}s^k \prod_\rho (1 - s/\rho)e^{s/\rho}$ for some $a, b \in \mathbb{C}$ and $k \geq 0$.

The first source states (i) "when s stays away from negative reals", which is the sector condition; the second recalls (ii); the third proves (iii) in the form of Hadamard's factorization theorem. The count of zeros of ζ_K [HSW21, Theorem 1.1, p. 2] gives, for the number $N_K(T)$ of zeros ρ of ξ_K with $|\operatorname{Im}\rho| \leq T$, the bounds $N_K(T) = O(T \log T)$ and $N_K(T+1) - N_K(T) = O(\log T)$.

By (ii) and (iii), the function

$$G_\xi(z) = -(\log \xi_K)''(z) = \sum_\rho \frac{m(\rho)}{(z-\rho)^2} \qquad (z \notin \operatorname{div} \xi_K), \tag{7.3}$$

with $m(\rho) = \operatorname{ord}_\rho \xi_K$, is given by an absolutely and locally uniformly convergent series.

Proposition 7.35. *Let K be a number field with Stirling constants A, B, C, E and standard branch f of $\log \xi_K$ (Definition 7.33(d),(e)), and let $s \in \mathbb{C}$. Then, for $z = s + x$ as $x \to +\infty$,*

$$f(z) = Az \log z + Bz + C \log z + E + O(|z|^{-1}), \qquad f'(z) = A \log z + A + B + \frac{C}{z} + O(|z|^{-2}),$$

and $G_\xi(z) = -f''(z) = -A/z + C/z^2 + O(|z|^{-3})$.

Proof. As $x \to \infty$, $\arg z \to 0$, so Theorem 7.34 (i) applies to z and $z/2$; the remainder R is holomorphic on the sector, and Cauchy's estimate on the disc of radius $|z|/4$ about z, which lies in a slightly larger sector, gives $R'(z) = O(|z|^{-2})$ and $R''(z) = O(|z|^{-3})$. Expanding, $-\frac{z}{2}\log\pi + \log\Gamma(\frac{z}{2}) = \frac{z}{2}\log z - \frac{z}{2}(1 + \log 2\pi) - \frac{1}{2}\log z + \frac{1}{2}\log 4\pi + O(|z|^{-1})$ and $\log 2 - z\log 2\pi + \log\Gamma(z) = z\log z - z(1+\log 2\pi) - \frac{1}{2}\log z + \frac{1}{2}\log 8\pi + O(|z|^{-1})$, with derivatives of the remainders $O(|z|^{-2})$ and $O(|z|^{-3})$; and $\log z + \log(z-1) = 2\log z + O(|z|^{-1})$, likewise. Each rational prime p lies below at most n prime ideals, each of norm a power p^f with $1 \leq f \leq n$, so the prime sum and its first two derivatives are bounded by a constant multiple of $\sum_{m \geq 2} (1 + \log m)^2 m^{-\operatorname{Re} z}$, which decays exponentially in $\operatorname{Re} z$ and is therefore $O(|z|^{-3})$. Adding up gives the expansion of f with the stated A, B, C, E, and differentiating it termwise gives those of f' and f''. □

Proposition 7.36. *Let $a \in \mathbb{C} \setminus (-\infty, 0]$ and $0 < \operatorname{Re} w < 2$. Then, with the principal power,*

$$a^{-w} = P(w) \int_0^\infty x^{1-w}(x+a)^{-2}\, dx, \qquad P(w) = \frac{\sin \pi w}{\pi(1-w)}.$$

The function P is entire, $P(1) = 1$, and $(2\pi)^w P(w) = w + (1 + \log 2\pi)w^2 + O(w^3)$ at $w = 0$.

Proof. For $a > 0$ the substitution $x = at$ gives $a^{-w}B(2-w, w) = a^{-w}\Gamma(2-w)\Gamma(w) = a^{-w}(1-w)\pi/\sin \pi w$. For a in a compact subset of $\mathbb{C} \setminus (-\infty, 0]$ one has $|x + a| \geq c(1 + x)$ for $x \geq 0$, so the integral is holomorphic in a and the identity persists. The function P has a removable singularity at $w = 1$ with value 1, and $P(w) = w + w^2 + O(w^3)$. □

Theorem 7.37 ((ii) Soulé; Illies; [Den94, Theorem 7.18(ii), p. 35])**.** *Let K be a number field with $n = [K : \mathbb{Q}]$, r_1 real and r_2 complex places, and discriminant D_K, and let (V, Θ) be a spectral core with spectrum $\operatorname{div} \xi_K$, counted with multiplicity.*

(i) For $s \in U_{\mathrm{pr}}$ the spectral zeta function $Z(w; s)$ of the principal cut converges absolutely for $\operatorname{Re} w > 1$ and continues holomorphically to $\{\operatorname{Re} w > -1\} \setminus \{1\}$, with a simple pole at $w = 1$ of residue $-\pi n$, and $Z(0; s) = \frac{n}{2}s + 2 - \frac{r_1 + r_2}{2}$.

(ii) (V, Θ) is zeta-regularizable for its principal cut, and

$$D_1(s) = \det{}_\infty\Big(\frac{s - \Theta}{2\pi}\Big) = 2^{-n/2}(2\pi)^{-2}|D_K|^{-s/2}\xi_K(s) = \frac{s}{2\pi} \cdot \frac{s-1}{2\pi} \cdot \widehat{\zeta}_K(s).$$

(iii) *Every Weil datum of type* $(1,0)$ *on* $\overline{X}_K$ *satisfying AW1 and AW2 for* $(\Lambda_K, \nVdash)$ *satisfies AW-det for* Λ_K *with the principal cuts and normalization term*

$$e^{a_K + b_K s} = 2^{-n/2} |D_K|^{-s/2}, \qquad a_K = -\tfrac{n}{2}\log 2, \quad b_K = -\tfrac{1}{2}\log|D_K|,$$

and satisfies AW-det for $\widehat{\zeta}_K$ *with normalization term* 1.

(iv) $D_1(1-s) = |D_K|^{s-1/2} D_1(s)$.

Proof of (i). Let $s \in U_{\mathrm{pr}}$, put $g_s(x) = G_\xi(s+x)$ for $x \ge 0$ (the ray $s + [0,\infty)$ misses the zeros), and let $1 < u = \operatorname{Re} w < 2$. For a zero ρ put $a = s - \rho \notin (-\infty, 0]$. For $x \ge 2|a|$ one has $|x + a| \ge x/2$, so $\int_{2|a|}^{\infty} x^{1-u}|x+a|^{-2}dx \le \frac{4}{u}(2|a|)^{-u}$. Since $0 \le \operatorname{Re}\rho \le 1$, all but finitely many zeros satisfy $|\operatorname{Im} a| \ge |a|/4$, and then $\int_0^{2|a|} x^{1-u}|x+a|^{-2}dx \le 16|a|^{-2}(2|a|)^{2-u}/(2-u)$. Hence $\sum_\rho m(\rho)\int_0^\infty x^{1-u}|x+s-\rho|^{-2}dx$ is bounded by a constant times $\sum_\rho m(\rho)|s-\rho|^{-u}$, which is finite because $N_K(T) = O(T\log T)$. The same bound gives absolute convergence of $Z(w;s)$ for $\operatorname{Re} w > 1$. Since dividing by 2π does not change arguments, $((s-\rho)/2\pi)^{-w} = (2\pi)^w (s-\rho)^{-w}$, and Proposition 7.36, Fubini's theorem, and (7.3) give, for $1 < \operatorname{Re} w < 2$,

$$Z(w;s) = (2\pi)^w P(w)\, I(w), \qquad I(w) = \int_0^\infty x^{1-w} g_s(x)\, dx.$$

By Proposition 7.35, $g_s(x) = c_1 x^{-1} + c_2 x^{-2} + O(x^{-3})$ as $x \to \infty$ with $c_1 = -A$ and $c_2 = As + C$, and g_s is continuous on $[0,\infty)$. Hence, for $1 < \operatorname{Re} w < 2$,

$$I(w) = \int_0^1 x^{1-w} g_s\, dx + \int_1^\infty x^{1-w}\Big(g_s - \frac{c_1}{x} - \frac{c_2}{x^2}\Big)dx + \frac{c_1}{w-1} + \frac{c_2}{w},$$

where the first integral is holomorphic for $\operatorname{Re} w < 2$ and the second for $\operatorname{Re} w > -1$. So I continues meromorphically to $-1 < \operatorname{Re} w < 2$ with simple poles at $w = 1$ and $w = 0$, of residues c_1 and c_2. The factor $(2\pi)^w P(w)$ is entire, vanishes at $w = 0$, and equals 2π at $w = 1$ (Proposition 7.36). Together with the absolutely convergent series for $\operatorname{Re} w > 1$, this gives the continuation of Z, the residue $2\pi c_1 = -\pi n$ at $w = 1$, and $Z(0;s) = c_2$. For later use we record, from the expansions at $w = 0$,

$$\partial_w Z(0;s) = (1 + \log 2\pi)c_2 + I_0, \qquad I_0 = \lim_{X\to\infty}\Big[\int_0^X x\, g_s(x)\, dx - c_1 X - c_2 \log X\Big], \tag{7.4}$$

where I_0 is the constant term of I at $w = 0$: it equals $\int_0^1 x g_s\, dx + \int_1^\infty x(g_s - c_1 x^{-1} - c_2 x^{-2})\, dx - c_1$, which is the limit displayed. □

Proof of (ii). Extend the standard branch f (Definition 7.33(e)) to U_{pr} by $f(s) = f(s+X) - \int_0^X (\xi_K'/\xi_K)(s+x)\, dx$ for any X with $\operatorname{Re} s + X > 1$. The value does not depend on X; since the segment $s + [0, X]$ stays away from the discrete set of zeros for s in a small disc, f is holomorphic on U_{pr}, with $f' = \xi_K'/\xi_K$, $f'' = -G_\xi$, and $e^f = \xi_K$. Integrating by parts, $\int_0^X x\, g_s(x)\, dx = -X f'(s+X) + f(s+X) - f(s)$. With $z = s + X$, Proposition 7.35 gives

$$-X f'(z) + f(z) = As\log z + Bs + C\log z + E - C - AX + o(1),$$

using $z - X = s$ and $X/z \to 1$. Subtracting $c_1 X + c_2 \log X = -AX + (As + C)\log X$ and letting $X \to \infty$ gives $I_0 = Bs + E - C - f(s)$, so by (7.4)

$$\partial_w Z(0;s) = (1+\log 2\pi)(As + C) + Bs + E - C - f(s) = \tfrac{s}{2}\log|D_K| + \tfrac{n}{2}\log 2 + 2\log 2\pi - f(s),$$

because $A(1 + \log 2\pi) + B = \frac{1}{2}\log|D_K|$ and $C\log 2\pi + E = 2\log 2\pi + (\frac{r_1}{2} + r_2)\log 2$. The right side is holomorphic on U_{pr}, and $\exp(-\partial_w Z(0;s)) = 2^{-n/2}(2\pi)^{-2}|D_K|^{-s/2}\xi_K(s)$ is entire. This is zeta-regularizability (Definition 5.2) and the first formula; the second is (7.2) with $\xi_K = s(s-1)\Lambda_K$. □

Proof of (iii). By AW1, $V^0 \cong \mathbb{1}$ and $V^2 \cong \mathbb{1}(-1)$ are lines with generators 0 and 1, so $D_0 = s/2\pi$ and $D_2 = (s-1)/2\pi$ for every cut, and $\mathrm{Sp}_0 = [0]$, $\mathrm{Sp}_2 = [1]$. By AW2, $\mathrm{Sp}_1 = \operatorname{div}\Lambda_K + [0] + [1] = \operatorname{div}\xi_K$, so (ii) applies to V^1, each D_j has divisor Sp_j, and $D_1/(D_0D_2) = 4\pi^2 D_1/(s(s-1)) = 2^{-n/2}|D_K|^{-s/2}\Lambda_K = \widehat{\zeta}_K$. □

Proof of (iv). By (ii) and $\xi_K(1-s) = \xi_K(s)$ (Proposition 3.6). □

For $K = \mathbb{Q}$, Theorem 7.37(ii) is Theorem 7.32. For general K, the formula $D_1 = \frac{s}{2\pi}\frac{s-1}{2\pi}\widehat{\zeta}_K$ is the case of the trivial character of [Den94, Theorem 7.18(ii), p. 35], which the source attributes to Soulé and to the Diplomarbeit of Illies (Münster, 1992). It is natural to expect that computing the normalization needs the general theory of regularized products; in fact the proof above uses only Hadamard's factorization, Stirling's formula, and the zero count (Theorem 7.34 and the remarks after it), and a beta integral (Proposition 7.36). The constant a_K is the regularized determinant of the archimedean spectra: the spectra $\{-2\nu\}_{\nu\geq 0}$ and $\{-\nu\}_{\nu\geq 0}$ have regularized products $\prod_\nu\big(\frac{\nu}{\pi}+\frac{s}{2\pi}\big) = \Gamma_{\mathbb{R}}^{\mathrm{D}}(s)^{-1}$ and $\prod_\nu\big(\frac{\nu}{2\pi}+\frac{s}{2\pi}\big) = \Gamma_{\mathbb{C}}^{\mathrm{D}}(s)^{-1}$ [Den91, Proposition 2.1, p. 248], and $a_K = r_1\log(\Gamma_{\mathbb{R}}^{\mathrm{D}}/\Gamma_{\mathbb{R}}) + r_2\log(\Gamma_{\mathbb{C}}^{\mathrm{D}}/\Gamma_{\mathbb{C}})$.

These notes keep the completion Λ_K of Definition 1.16. With it, the normalization term of AW-det on $\overline{X}_K$ is the explicit function $2^{-n/2}|D_K|^{-s/2}$ of Theorem 7.37(iii); with Deninger's completion $\widehat{\zeta}_K$ it is 1, exactly as over $\mathbb{F}_q$ (Theorem 7.24(ii)).

It is natural to expect that for every number field the regularized determinant of the nontrivial zeros reproduces the completion Λ_K of Definition 1.16, that is, $D_1 = \xi_K/4\pi^2$ with normalization term 1 relative to Λ_K (for $K = \mathbb{Q}$, $D_1 = \xi/2\pi^2$); in fact the normalization term relative to Λ_K is $2^{-n/2}|D_K|^{-s/2}$, which is not identically 1 for any K, and the normalization term is 1 exactly relative to $\widehat{\zeta}_K$, which has Deninger's gamma factors and no discriminant factor (Theorem 7.37).

Theorem 7.38 (after [Den94, Lemma (1.7), p. 6, and (7.19.1), p. 36])**.** *Let K and (V,Θ) be as in Theorem 7.37. Then (V,Θ) is zeta-regularizable for its reflected cut (Definition 5.3); for $s \in U_{\mathrm{rf}}$ its spectral zeta function is $Z_{\mathrm{rf}}(w;s) = e^{-i\pi w}Z(w;1-s)$, with Z that of the principal cut; and its regularized determinant is*

$$D_{\mathrm{rf}}(s) = i^{r_2}e^{-i\pi ns/2}\,2^{-n/2}(2\pi)^{-2}|D_K|^{(s-1)/2}\,\xi_K(s) = e^{a'+b's}D_1(s),$$

with $a' = \frac{i\pi r_2}{2} - \frac{1}{2}\log|D_K|$ and $b' = \log|D_K| - \frac{i\pi n}{2}$.

Proof. For $u \notin [0,\infty)$ the argument of u in $[0,2\pi)$ is $\mathrm{Arg}(-u)+\pi$. Hence, for $s \in U_{\mathrm{rf}}$, $((s-\rho)/2\pi)^{-w} = e^{-i\pi w}(((1-s)-(1-\rho))/2\pi)^{-w}$ with the reflected power on the left and the principal power on the right. As $\rho \mapsto 1-\rho$ permutes the zeros with multiplicity and $1-s \in U_{\mathrm{pr}}$ exactly when $s \in U_{\mathrm{rf}}$, summing gives $Z_{\mathrm{rf}}(w;s) = e^{-i\pi w}Z(w;1-s)$, which is regular at $w=0$ by Theorem 7.37(i). Then $-\partial_w Z_{\mathrm{rf}}(0;s) = i\pi Z(0;1-s) - \partial_w Z(0;1-s)$, so $D_{\mathrm{rf}}(s) = e^{i\pi Z(0;1-s)}D_1(1-s)$. By Theorem 7.37(i), $e^{i\pi Z(0;1-s)} = e^{i\pi(n-r_1-r_2)/2}e^{2\pi i}e^{-i\pi ns/2} = i^{r_2}e^{-i\pi ns/2}$, and by (ii) and (iv) of the same theorem $D_1(1-s) = 2^{-n/2}(2\pi)^{-2}|D_K|^{(s-1)/2}\xi_K(s)$. The quotient $D_{\mathrm{rf}}/D_1 = i^{r_2}|D_K|^{s-1/2}e^{-i\pi ns/2}$ is $e^{a'+b's}$. □

The cut dependence has the shape of Proposition 7.21: the real part of the slope b' is the logarithm of the conductor, $\log|D_K|$ in place of $\log q$, and the phase $e^{-i\pi ns/2}$ has no finite-field counterpart because the archimedean part of the spectrum is not periodic.

It is natural to expect that the half-plane pieces $Z_\pm$ of the spectral zeta function are regular at $w=0$, so that $D_1(1-s)$ and $D_1(s)$ can be compared piece by piece through the phases $e^{\pm i\pi w}$; in fact each piece has a pole at $w=0$ for every $s \neq \frac{1}{2}$, with the residue that the main term $n\log T$ of the counting function predicts, and the poles cancel in the sum. It is then natural to expect that the normalization over a number field cannot be determined without a

finer regularization of the pieces; in fact the full spectral zeta function is regular at $w = 0$ and determines it (Theorem 7.37).

Proposition 7.39 (after [Den94, (1.3)–(1.7), pp. 4–6, and Theorem (7.20), p. 37])**.** *Let K and (V, Θ) be as in Theorem 7.37, and let $s \in \mathbb{C}$ with $\operatorname{Im} s \neq \operatorname{Im} \rho$ for every zero ρ of ξ_K. Then the half-plane pieces $Z_\pm(w; s)$ of Definition 7.33(c) converge absolutely for $\operatorname{Re} w > 1$, $Z_+ + Z_- = Z$, and each continues meromorphically to $\operatorname{Re} w > -1$ with poles only at $w = 0$ and $w = 1$:*

$$\operatorname{Res}_{w=0} Z_\pm(w; s) = \mp \frac{i\,n\,(s - \frac{1}{2})}{2\pi}, \qquad Z_\pm(w; s) = \pm \frac{i\,n}{(w-1)^2} + O\big((w-1)^{-1}\big) \quad (w \to 1).$$

In particular neither piece is regular at $w = 0$ unless $s = \frac{1}{2}$. With the opposite labelling (the piece over the zeros below s called Z_+) the signs of both residues change.

Proof. Absolute convergence and $Z_+ + Z_- = Z$ follow from Theorem 7.37(i), since no zero has $\operatorname{Im} \rho = \operatorname{Im} s$. For a zero above s, $s - \rho$ lies in the lower half plane and its argument in $[0, 2\pi)$ is its principal argument plus 2π; for a zero below s the two arguments agree. Hence, for $\operatorname{Re} w > 1$, the reflected spectral zeta function is $Z_{\mathrm{rf}}(w; s) = e^{-2\pi i w} Z_+(w; s) + Z_-(w; s)$, and by Theorem 7.38

$$(1 - e^{-2\pi i w})\, Z_+(w; s) = Z(w; s) - e^{-i\pi w} Z(w; 1 - s).$$

Both s and $1 - s$ lie in U_{pr}, so by Theorem 7.37(i) the right side, $N(w)$, is holomorphic on $\{\operatorname{Re} w > -1\} \setminus \{1\}$. The factor $1 - e^{-2\pi i w}$ vanishes on $\operatorname{Re} w > -1$ only at $w = 0, 1, 2, \dots$, simply; for $\operatorname{Re} w > 1$, Z_+ is holomorphic, so the only possible poles are at 0 and 1. At $w = 0$, $N(0) = Z(0; s) - Z(0; 1 - s) = n(s - \frac{1}{2})$ and $1 - e^{-2\pi i w} = 2\pi i w + O(w^2)$, so the residue is $n(s - \frac{1}{2})/2\pi i$. At $w = 1$, $N(w) = -2\pi n/(w - 1) + O(1)$ because $e^{-i\pi w} = -1 + O(w - 1)$ and both $Z(\cdot; s)$ and $Z(\cdot; 1 - s)$ have residue $-\pi n$, while $1 - e^{-2\pi i w} = 2\pi i(w - 1) + O((w - 1)^2)$; so $Z_+ = in/(w - 1)^2 + O((w - 1)^{-1})$. The statements for $Z_- = Z - Z_+$ follow, Z being regular at 0 and having a simple pole at 1. □

It is natural to expect that zeta-regularizability of $\operatorname{div} \xi_K$ follows from its counting function alone, which has an explicit main term and error $O(\log T)$; in fact an error of that size can create a pole at $w = 0$, and the proof of Theorem 7.37 uses the Hadamard product and the Stirling asymptotics of ξ_K along horizontal rays, not only the count. It is equally natural to expect that for the zeros of ξ_K the fluctuation of the count contributes to the residues of $Z_\pm$ at $w = 0$; in fact it contributes nothing, the residues being those predicted by the main term (Proposition 7.39).

Proposition 7.40 (compare [Den92, Remark (3.4)(2), p. 147])**.** *Let K be a number field, let $\Sigma = \operatorname{div} \xi_K$ and*

$$\Sigma' = \Sigma + \sum_{k \geq 1} \big([\tfrac{1}{2} + ie^k] + [\tfrac{1}{2} - ie^k] \big).$$

(i) Σ' is the spectrum, with multiplicity, of a spectral core; it is invariant under $\rho \mapsto 1 - \rho$ and $\rho \mapsto \bar{\rho}$; and its number of points with $0 < \operatorname{Im} \rho \leq T$ exceeds that of Σ by $\lfloor \log T \rfloor$ for $T \geq 1$.

(ii) For every s in the principal domain U_{pr} of Σ', the spectral zeta function of Σ' for the principal cut continues meromorphically to $\operatorname{Re} w > -1$ and has a simple pole at $w = 0$ with residue 2. So Σ' is not zeta-regularizable for its principal cut.

Proof of (i). The added points are discrete, symmetric under both maps, and $\#\{k \geq 1 : e^k \leq T\} = \lfloor \log T \rfloor$; take the direct sum of the spectral cores with spectra Σ and the added points. □

Proof of (ii). The spectral zeta function of Σ' is that of Σ, which is regular at $w = 0$ by Theorem 7.37(i), plus $Y(w) = \sum_k \big[((s - \frac{1}{2} - ie^k)/2\pi)^{-w} + ((s - \frac{1}{2} + ie^k)/2\pi)^{-w} \big]$. Put $\varepsilon_k = (s - \frac{1}{2})e^{-k}$ and let k_0 satisfy $|\varepsilon_k| < \frac{1}{2}$ for $k \geq k_0$. For such k, $s - \frac{1}{2} \mp ie^k = e^k(\mp i)(1 \pm i\varepsilon_k)$, and the principal

logarithms add because $|\operatorname{Arg}(1\pm i\varepsilon_k)|<\pi/6$; so the k-th term is $(2\pi)^w e^{-kw}\big[e^{i\pi w/2}(1+i\varepsilon_k)^{-w}+e^{-i\pi w/2}(1-i\varepsilon_k)^{-w}\big]$. Since $|(1\pm i\varepsilon)^{-w}-1|\le c|w||\varepsilon|$ for w bounded and $|\varepsilon|<\frac12$, the sum over $k\ge k_0$ equals

$$\frac{2(2\pi)^w\cos(\pi w/2)\,e^{-k_0 w}}{1-e^{-w}}+h(w),$$

where h is a sum of terms $O(|w|e^{-k(\operatorname{Re}w+1)})$, holomorphic on $\operatorname{Re}w>-1$. The first term has a simple pole at $w=0$ with residue 2, and the finitely many terms with $k<k_0$ are entire. So Y, and with it the spectral zeta function of Σ', has a simple pole at $w=0$ of residue 2, and Definition 5.2(c) fails. □

7.8. **The trace-formula axiom.** Over $\mathbb{F}_q$ the cohomology is constructed independently of the zeta function, the Lefschetz trace formula for all powers of Frobenius is a theorem about it, and rationality with the correct multiplicities is a consequence of that formula. This subsection compares the divisor axiom AW2 on $\overline{X}_K$ with the trace-formula axiom AW6′ of Definition 5.9, and calibrates the comparison against $\mathbb{F}_q$. The input on the number-field side is the explicit formula.

Theorem 7.41 (Weil; [Den01, Theorem 4.1, (17), p. 15], [Den02, (1), p. 1])**.** *Let K be a number field and $\varphi\in C_c^\infty(\mathbb{R})$, with transform Φ and geometric side $G_K(\varphi)$ (Definitions 5.7 and 5.8). Then*

$$\Phi(0)-\sum_\rho \operatorname{ord}_\rho(\xi_K)\,\Phi(\rho)+\Phi(1)=G_K(\varphi),$$

the sum running over the distinct zeros ρ of ξ_K and converging absolutely.

The first source states this identity for every K, with the archimedean terms given explicitly for φ supported in $(0,\infty)$ or in $(-\infty,0)$, where they agree with Definition 5.8; it notes that they can be written on all of $\mathbb{R}$ in different ways and refers to Barner [Bar81] for a proof. The explicit formula in this generality is due to Weil [Wei52], as reported in [Bom06, §V, p. 8]. The second states it for $K=\mathbb{Q}$ with the archimedean term $W_{\mathbb{R}}$ of Definition 5.8 on all of $\mathbb{R}$. For general K the form of $W_{\mathbb{R}}$ and $W_{\mathbb{C}}$ at $t=0$ is the one produced by the contour argument, which we recall.

Derivation of the archimedean terms. Put $\lambda=\Lambda_K'/\Lambda_K=\xi_K'/\xi_K-1/s-1/(s-1)$ and fix $c>1$. Integration by parts gives $|\Phi(s)|\le C_k(1+|s|)^{-k}$ on every vertical strip, for every k. By Theorem 7.34(ii) and (iii), $\xi_K'/\xi_K(s)=b+\sum_\rho(1/(s-\rho)+1/\rho)$ with $\sum|\rho|^{-2}<\infty$, the zeros repeated with multiplicity. Since $N_K(T+2)-N_K(T-2)=O(\log T)$, each interval $[j,j+1]$ contains a point T_j at distance at least $1/(C\log j)$ from the imaginary part of every zero. For $s=\sigma+iT_j$ with $1-c\le\sigma\le c$, the zeros with $|\rho|>2|s|$ contribute $O(|s|\sum|\rho|^{-2})$, and the $O(|s|\log|s|)$ others contribute at most $|s|\cdot C\log j/\min|\rho|$ each; so $|\lambda(s)|=O(T_j^3)$, and likewise at $-T_j$ because the zeros are symmetric under conjugation. The residue theorem on the rectangles with vertices $1-c\pm iT_j$, $c\pm iT_j$ therefore gives, in the limit, since the residue of $\Phi\lambda$ at 0 and 1 is $-\Phi(0)$, $-\Phi(1)$ and at ρ is $\operatorname{ord}_\rho(\xi_K)\Phi(\rho)$,

$$\frac{1}{2\pi i}\Big[\int_{(c)}-\int_{(1-c)}\Big]\Phi(s)\lambda(s)\,ds=-\Phi(0)+\sum_\rho\operatorname{ord}_\rho(\xi_K)\Phi(\rho)-\Phi(1).$$

On $\operatorname{Re}s=1-c$ substitute $s=1-u$ and use $\lambda(1-u)=-\lambda(u)$, which follows from $\Lambda_K(1-s)=\Lambda_K(s)$ (Theorem 3.5); this gives

$$\Phi(0)-\sum_\rho\operatorname{ord}_\rho(\xi_K)\Phi(\rho)+\Phi(1)=-\frac{1}{2\pi i}\int_{(c)}\big[\Phi(s)+\Phi(1-s)\big]\lambda(s)\,ds.$$

For $\operatorname{Re} s > 1$, $\lambda(s) = \frac{1}{2}\log|D_K| + r_1\Gamma_{\mathbb{R}}'/\Gamma_{\mathbb{R}}(s) + r_2\Gamma_{\mathbb{C}}'/\Gamma_{\mathbb{C}}(s) - \sum_{\mathfrak{p}}\sum_{k\geq 1}\log N\mathfrak{p}\, N\mathfrak{p}^{-ks}$, and Fourier inversion on $\operatorname{Re} s = c$ gives $\frac{1}{2\pi i}\int_{(c)}\Phi(s)e^{-sT}ds = \varphi(T)$ and $\frac{1}{2\pi i}\int_{(c)}\Phi(1-s)e^{-sT}ds = e^{-T}\varphi(-T)$. The constant term gives $-\log|D_K|\,\varphi(0)$ and the prime terms give the prime sum of G_K (the interchange is justified by $|\Phi(c+iy)| \leq C(1+|y|)^{-2}$). For the archimedean terms insert Gauss's integral $\psi(s) = \Gamma'/\Gamma(s) = \int_0^\infty \big(e^{-u}/u - e^{-su}/(1-e^{-u})\big)du$ ($\operatorname{Re} s > 0$): with $u = 2t$, $\Gamma_{\mathbb{R}}'/\Gamma_{\mathbb{R}}(s) = -\frac{1}{2}\log\pi + \int_0^\infty \big(e^{-2t}/2t - e^{-st}/(1-e^{-2t})\big)dt$, and $\Gamma_{\mathbb{C}}'/\Gamma_{\mathbb{C}}(s) = -\log 2\pi + \psi(s)$. Interchanging the s- and t-integrals (Fubini: the t-integral of the absolute value grows at most linearly in $|s|$ on $\operatorname{Re} s = c$, and Φ decays faster than any power there), the term $-e^{-st}/(1-e^{-2t})$ produces $(\varphi(t)+e^{-t}\varphi(-t))/(1-e^{-2t})$, the term $e^{-2t}/2t$ produces $-e^{-2t}\varphi(0)/t$, and the constant produces $\log\pi\cdot\varphi(0)$; this is $W_{\mathbb{R}}$, and $W_{\mathbb{C}}$ follows in the same way. □

The derivation uses Gauss's integral for ψ, a classical consequence of the product formula for Γ, which is not re-derived here. For $\varphi \in \mathcal{S}^+$ the theorem is the one-sided explicit formula of [Den01, (18), p. 15] with (5.1), and no choice at $t = 0$ enters.

Proposition 7.42. *Let $\Sigma \subset \{a \leq \operatorname{Re} s \leq b\}$ be a discrete set with $\#\{\rho \in \Sigma : |\rho| \leq R\} = O(R^B)$, and let $c : \Sigma \to \mathbb{C}$ satisfy $|c_\rho| \leq C(1+|\rho|)^A$.*

(i) For every $\varphi \in C_c^\infty(\mathbb{R})$ the series $\sum_\rho c_\rho\Phi(\rho)$ converges absolutely, and $\varphi \mapsto \sum_\rho c_\rho\Phi(\rho)$ is a distribution T_c on $\mathbb{R}$.

(ii) If, for some $T_0 \in \mathbb{R}$, $T_c(\varphi) = 0$ for every φ in $C_c^\infty((T_0,\infty))$, or for every φ in $C_c^\infty((-\infty,T_0))$, then $c = 0$.

Proof of (i). Fix a real $s_0 > b$ and an integer $N > A + B$. Partial summation against the counting function gives $\sum_\rho |c_\rho||\rho - s_0|^{-N} < \infty$. Put $g(t) = \sum_\rho c_\rho(\rho - s_0)^{-N}e^{\rho t}$; since $|e^{\rho t}| \leq e^{\max(|a|,|b|)|t|}$, the series converges uniformly on compact sets, so g is continuous. For $\varphi \in C_c^\infty(\mathbb{R})$, N integrations by parts give $\int e^{\rho t}(-\frac{d}{dt} - s_0)^N\varphi\,dt = (\rho - s_0)^N\Phi(\rho)$, hence $|\Phi(\rho)| \leq C_\varphi|\rho - s_0|^{-N}$ on the strip, which gives absolute convergence, and $\sum_\rho c_\rho\Phi(\rho) = \int g\,(-\frac{d}{dt} - s_0)^N\varphi\,dt$, which is continuous in φ. So $T_c = (\frac{d}{dt} - s_0)^N g$ in the sense of distributions. □

Proof of (ii). Suppose $T_c = 0$ on (T_0,∞). Then $(\frac{d}{dt} - s_0)^N g = 0$ there, so $g(t) = e^{s_0 t}P(t)$ on (T_0,∞) for a polynomial P of degree less than N (the distributional solutions of a linear differential equation with constant coefficients are classical). For $\operatorname{Re} s > s_0$ compute $\mathcal{L}(s) = \int_{T_0}^\infty g(t)e^{-st}\,dt$ in two ways. Termwise, which is justified by $\sum_\rho |c_\rho||\rho - s_0|^{-N}\int_{T_0}^\infty e^{(b-\operatorname{Re} s)t}dt < \infty$,

$$\mathcal{L}(s) = M(s) := \sum_\rho c_\rho(\rho - s_0)^{-N}\frac{e^{(\rho-s)T_0}}{s-\rho}.$$

From the polynomial form, $\mathcal{L}(s) = e^{-(s-s_0)T_0}Q(1/(s-s_0))$ with Q a polynomial, holomorphic on $\mathbb{C}\setminus\{s_0\}$. The series M converges locally uniformly on the complement of $\Sigma_c = \{\rho : c_\rho \neq 0\}$, because on a compact set D disjoint from Σ_c the factors $e^{(\rho-s)T_0}$ are bounded and $|s-\rho|^{-1} \leq C_D$; so M is meromorphic on $\mathbb{C}$ with simple poles exactly at the points of Σ_c, of residue $c_\rho(\rho - s_0)^{-N} \neq 0$. The set $\mathbb{C}\setminus(\Sigma_c \cup \{s_0\})$ is connected, and M agrees there with $e^{-(s-s_0)T_0}Q(1/(s-s_0))$ by the identity theorem, which has no pole in Σ_c (note $s_0 \notin \Sigma_c$). So Σ_c is empty. The case $(-\infty,T_0)$ follows by $t \mapsto -t$, $\rho \mapsto -\rho$. □

It is natural to expect that an exponential sum over a discrete spectrum determines its coefficients only if it vanishes for all positive times; in fact vanishing on one tail (T_0,∞) suffices (Proposition 7.42(ii)), while vanishing on a bounded interval does not (Theorem 7.50).

The trace formula of Definition 5.9 is the form in which Deninger expects the explicit formula to arise from a Lefschetz formula [Den98, (10), p. 172], [Den01, (19), p. 16].

Theorem 7.43. *Let K be a number field and let a Weil datum of type $(1,0)$ on $\overline{X}_K$ satisfy AW1 for $(\Lambda_K, \mathbb{1})$. The following are equivalent:*

(i) the datum satisfies AW6′;
(ii) it satisfies AW6′ on $\mathcal{S}^+$;
(iii) it satisfies AW6′ on $\mathcal{S}_{>T_0}$ for some $T_0 \in \mathbb{R}$;
(iv) the datum satisfies AW2 for $(\Lambda_K, \mathbb{1})$, that is, $\mathrm{Sp}_1 = \operatorname{div} \xi_K$.

In particular, for $T_0 \geq 0$, the identity (AW6′) for test functions supported in (T_0, ∞), whose geometric side involves only the prime powers with $k \log N\mathfrak{p} > T_0$ and the archimedean places, determines Sp_1 with multiplicities.

Proof. (i)⇒(ii)⇒(iii) is clear, since $\mathcal{S}^+ = \mathcal{S}_{>0}$. For (iii)⇒(iv): by AW1, $V^0 \cong \mathbb{1}$ and $V^2 \cong \mathbb{1}(-1)$, so $\mathrm{Sp}_0 = [0]$, $\mathrm{Sp}_2 = [1]$, and (AW6′) reads $\Phi(0) - \sum_\rho m_1(\rho)\Phi(\rho) + \Phi(1) = G_K(\varphi)$. Subtracting Theorem 7.41 gives $\sum_\rho c_\rho \Phi(\rho) = 0$ for $\varphi \in \mathcal{S}_{>T_0}$, with $c = m_1 - \operatorname{ord}(\xi_K)$ on $\Sigma = \operatorname{spec} \Theta_1 \cup \operatorname{div} \xi_K$. By (G) and $N_K(T) = O(T \log T)$, Σ lies in a vertical strip with polynomially bounded counting function, and $|c_\rho|$ is bounded by the sum of the two counting functions at $|\rho|$. By Proposition 7.42(ii), $c = 0$, that is, $\mathrm{Sp}_1 = \operatorname{div} \xi_K$; and $\operatorname{div} \Lambda_K = \operatorname{div} \xi_K - [0] - [1] = -\mathrm{Sp}_0 + \mathrm{Sp}_1 - \mathrm{Sp}_2$ is AW2. For (iv)⇒(i): V^0 and V^2 are lines, and $\operatorname{spec} \Theta_1 = \operatorname{div} \xi_K$ lies in $0 \leq \operatorname{Re} s \leq 1$ with counting function $O(R \log R)$, so (G) holds in every degree; the traces converge absolutely by Proposition 7.42(i), and the left side of (AW6′) is that of Theorem 7.41. □

It is natural to expect that, as over $\mathbb{F}_q$, imposing a trace-formula axiom in place of AW2 would make the divisor identity a theorem and the arithmetic axioms weaker; in fact, by Theorem 7.43, given AW1 the axiom AW6′ is equivalent to AW2. Imposing AW6′ instead of AW2 changes the form of the assumption, from "the flow has the right spectrum" to "the flow has the right trace formula", not its strength. The geometric side of AW6′ is built from the prime ideals, the archimedean places, and the discriminant alone, but no object satisfying it is known.

It is natural to expect that, as over $\mathbb{F}_q$, where $\log Z(X,t) = \sum_k N_k t^k/k$ is an identity of power series, the trace formula integrates to the determinant identity; in fact AW6′ determines the divisor only, and the determinant identity with its normalization term is a separate analytic statement about $\operatorname{div} \xi_K$: by Theorems 7.43 and 7.37(iii), a datum satisfying AW1 and AW6′ satisfies AW-det for Λ_K with normalization term $2^{-n/2}|D_K|^{-s/2}$.

For a datum of type $(d,0)$ with $d \geq 2$ the same argument shows that a trace formula determines only the alternating divisor $\sum_j (-1)^{j+1} \mathrm{Sp}_j$; it does not determine the degreewise spectra without positivity (Proposition 7.30, Theorem 7.28). For $d = 1$, AW1 fixes degrees 0 and 2, so no cancellation between degrees is possible.

The finite-field calibration is the following theorem.

Definition 7.44. Let X be a smooth projective geometrically connected variety over $\mathbb{F}_q$, with H^j and F as in Theorem 7.24; F is invertible on each H^j (Theorem 4.9). The *Lefschetz numbers* of X are $N_k = \sum_j (-1)^j \operatorname{Tr}(\mathrm{F}^k \mid H^j)$, $k \in \mathbb{Z}$.

Theorem 7.45 ((i) Grothendieck; (ii) after [Den02, (2), p. 3], [Den98, (10), p. 172])**.** *Let X be a smooth projective geometrically connected variety of dimension d over $\mathbb{F}_q$, with H^j, F, and the suspended datum $(\widehat{H}^\bullet, \Theta, \cup, \mathrm{Tr}, 0)$ as in Theorem 7.24, multiplicity functions m_j of $(\widehat{H}^j, \Theta_j)$, and Lefschetz numbers N_k (Definition 7.44).*

(i) $N_k = \#X(\mathbb{F}_{q^k})$ for $k \geq 1$, $N_0 = \sum_j (-1)^j \dim H^j$, and $N_{-k} = q^{-dk} N_k$.
(ii) For every $\varphi \in C_c^\infty(\mathbb{R})$,

$$\sum_{j=0}^{2d} (-1)^j \sum_\rho m_j(\rho)\Phi(\rho) = \log q \sum_{k \in \mathbb{Z}} N_k \, \varphi(k \log q),$$

the left side converging absolutely.

(iii) Let $(W^\bullet, \Theta', \cup, \mathrm{Tr}', *')$ be a Weil datum of type $(d,0)$ whose spectral cores satisfy (G), with spectral divisors Sp'_j. If the identity of (ii), with its right side, holds for its multiplicity functions for every $\varphi \in C_c^\infty((T_0,\infty))$ for some T_0, then $\sum_j (-1)^{j+1}\,\mathrm{Sp}'_j = \operatorname{div}\zeta_X$, that is, the datum satisfies AW2 for ζ_X.

Proof of (i). Take logarithms in the rationality statement of Theorem 3.8. Since $\log\det(1-\mathrm{F}t \mid H^j)^{-1} = \sum_k \mathrm{Tr}(\mathrm{F}^k \mid H^j)t^k/k$, this gives $\sum_k \#X(\mathbb{F}_{q^k})t^k/k = \sum_k N_k t^k/k$. The value N_0 is immediate. By Theorem 4.9, the pairing of H^j with H^{2d-j} is perfect and satisfies $\mathrm{tr}(\mathrm{F}u \cup \mathrm{F}v) = q^d\,\mathrm{tr}(u\cup v)$. So F on H^{2d-j} is q^d times the adjoint of F^{-1} on H^j, and hence $\mathrm{Tr}(\mathrm{F}^{-k} \mid H^j) = q^{-dk}\,\mathrm{Tr}(\mathrm{F}^k \mid H^{2d-j})$, and $(-1)^j = (-1)^{2d-j}$. □

Proof of (ii). By Definition 4.15 and Proposition 4.16(ii), $\operatorname{spec}\Theta_j$ is the union of the progressions $\Sigma_q(\alpha)$ over the eigenvalues α of F on H^j, with $m_j(\rho) = \dim H^j_{(\alpha)}$ for $\rho \in \Sigma_q(\alpha)$. Fix α and $\rho_0 \in \Sigma_q(\alpha)$, and put $L = \log q$ and $\psi(t) = \varphi(t)e^{\rho_0 t}$. Then $\Sigma_q(\alpha) = \{\rho_0 + 2\pi i n/L\}$, and the Poisson summation formula for the smooth compactly supported function ψ gives $\sum_n \Phi(\rho_0 + 2\pi i n/L) = \sum_n \int \psi(t)e^{2\pi i n t/L}dt = L\sum_k \psi(kL) = L\sum_k \alpha^k\varphi(kL)$, since $e^{\rho_0 kL} = \alpha^k$. Summing over α with multiplicity and over j with signs gives the identity; absolute convergence holds because Φ decays faster than any power on vertical strips. □

Proof of (iii). Subtract the identity (ii) for the suspended datum. The difference is $\sum_\rho c_\rho \Phi(\rho) = 0$ for $\varphi \in C_c^\infty((T_0,\infty))$, where c_ρ is the difference of the two alternating sums of multiplicities; both spectra lie in vertical strips with polynomially bounded counting functions. By Proposition 7.42(ii), $c = 0$, so $\sum_j (-1)^{j+1}\,\mathrm{Sp}'_j$ is the alternating spectral divisor of the suspended datum, which is $\operatorname{div}\zeta_X$ by Theorem 7.24(ii). □

So over $\mathbb{F}_q$ the Lefschetz formula for all powers of Frobenius from some point on determines the alternating divisor of ζ_X by the same uniqueness statement, Proposition 7.42, that gives Theorem 7.43; the difference between the two settings is that over $\mathbb{F}_q$ the formula is a theorem about an independently constructed cohomology. The atom at $t=0$ of the right side of (ii) is $\log q \cdot N_0$, an Euler characteristic; in the geometric side over K its place is taken by $-\log|D_K|$ and the archimedean constants.

It is natural to expect that a faithful analogue of the Lefschetz formula must be one-sided, since only the numbers N_k with $k \geq 1$ count points; in fact N_0 and N_{-k} are fixed by the Euler characteristic and by duality, so the suspended datum satisfies the two-sided identity of Theorem 7.45, and given AW1 the one-sided and two-sided forms of AW6′ are equivalent (Theorem 7.43).

The next statements concern the Hilbert form of AW6′.

Proposition 7.46. *Let a Weil datum of type $(d,0)$ have positive definite star form h_j in degree j and satisfy the condition of AW5 for this j.*

(i) h_j is a positive infinitesimal polarization of weight j on (V^j,Θ_j), so the datum satisfies AW4-V in degree j.

(ii) Θ_j is diagonalizable on V^j, every eigenvalue has real part $j/2$, and distinct eigenspaces are h_j-orthogonal.

(iii) T_j is the diagonal operator of an orthonormal basis of $\mathscr{H}^j$ consisting of eigenvectors of Θ_j in V^j; its spectrum is pure point, equal to $\{(\rho - j/2)/i : \rho \in \operatorname{spec}\Theta_j\}$ with the multiplicities m_j, without finite accumulation point.

(iv) If (V^j,Θ_j) satisfies (G), then for every $\varphi \in C_c^\infty(\mathbb{R})$ the operator $\int_{\mathbb{R}} \varphi(t)e^{jt/2}U_j(t)\,dt$ is of trace class, with trace $\sum_\rho m_j(\rho)\Phi(\rho)$.

Proof of (i). Each $v \in V^j$ lies in a finite-dimensional Θ_j-stable subspace, on which $U_j(t)v = e^{-jt/2}e^{t\Theta_j}v$ is smooth in t. Differentiating $h_j(U_j(t)v, U_j(t)v) = h_j(v,v)$ at $t=0$ gives $h_j(\Theta v, v) + h_j(v, \Theta v) = j h_j(v,v)$. Both sides are the diagonal values of Hermitian forms, so they agree as forms, which is (1.1). □

Proof of (ii). Proposition 3.3, by (i). □

Proof of (iii). By (i) and Proposition 6.5 there is an orthonormal basis (e_k) of $\mathscr{H}^j$ in V^j with $\Theta_j e_k = (j/2 + i\tau_k)e_k$, and $D = \operatorname{diag}(\tau_k)$ is self-adjoint with the stated spectrum. On V^j, which is the span of the e_k, the generator A_j of U_j acts as $\Theta_j - j/2$, so $T_j = -iA_j$ agrees with D there. Since V^j is a core for A_j, T_j is the closure of its restriction to V^j; the closure of the restriction of D to finite linear combinations of the e_k is D (truncate the expansion of a vector in its domain). So $T_j = D$. □

Proof of (iv). By (iii), $U_j(t)e_k = e^{it\tau_k}e_k$, so the operator maps e_k to $\Phi(j/2 + i\tau_k)e_k = \Phi(\rho_k)e_k$. A bounded operator that is diagonal in an orthonormal basis with absolutely summable entries is of trace class with trace their sum, and the entries are summable by Proposition 7.42 (i) with $c = m_j$. □

It is natural to expect that the requirement of AW6 that the completion lose no spectrum is an axiom independent of AW5; in fact it follows from AW5 (Proposition 7.46(iii)), and what AW6 adds to AW5 is exactly (G) and the trace formula, that is, AW6′. It is also natural to expect AW5 to be a purely analytic completeness condition, so that "AW1, AW5, and AW6′ imply AW2" would be the analogue of "Lefschetz implies rationality"; in fact AW5 presupposes a positive definite form h_j and, with its unitary flow, yields the positivity axiom in the form AW4-V (Proposition 7.46(i)), so these hypotheses imply the Riemann hypothesis for ζ_K, while over $\mathbb{F}_q$ rationality needs no polarization. The analogue of "Lefschetz implies rationality" is Theorem 7.43.

Theorem 7.47. *Let K be a number field and let a Weil datum of type $(1,0)$ on $\overline{X}_K$ satisfy AW1 for $(\Lambda_K, \nVdash)$, have positive definite star forms, and satisfy AW5.*

- *(i) For every $\mathcal{S}' \subset \mathcal{S}$, the datum satisfies AW6′ in Hilbert form on $\mathcal{S}'$ if and only if it satisfies AW6′ on $\mathcal{S}'$.*
- *(ii) If it satisfies AW6′ on $\mathcal{S}$, on $\mathcal{S}^+$, or on a tail class, then $\mathrm{Sp}_1 = \operatorname{div}\xi_K$, every zero of ξ_K has real part $\frac{1}{2}$, and $\dim\ker(\Theta_1 - \rho) = \operatorname{ord}_\rho \xi_K$ for every ρ.*
- *(iii) If every zero of ξ_K has real part $\frac{1}{2}$, the trivial model with spectrum $\operatorname{div}\xi_K$ (Definition 7.5) has positive definite star forms and satisfies AW1, AW3, AW4, AW5, and AW6′ in both forms.*

In particular, some Weil datum on $\overline{X}_K$ with positive definite star forms satisfies AW1, AW5, and AW6′ if and only if every zero of ξ_K has real part $\frac{1}{2}$.

Proof of (i). Both forms include (G), and by Proposition 7.46(iv) with $w = 0$ the traces $\operatorname{tr} A_\varphi^{(j)}$ are the distributional traces. □

Proof of (ii). By Theorem 7.43, $\mathrm{Sp}_1 = \operatorname{div}\xi_K$; by Proposition 7.46(ii), Θ_1 is diagonalizable with every eigenvalue on $\operatorname{Re} s = \frac{1}{2}$. □

Proof of (iii). By Propositions 7.6 and 7.7 (i), the trivial model satisfies AW1, AW2, AW3, and AW4, and its star forms are the standard positive forms in the given bases; h_1 is a positive infinitesimal polarization of weight one because Θ_1 is diagonal in an h_1-orthonormal basis with eigenvalues of real part $\frac{1}{2}$. By Proposition 6.5, $T = \operatorname{diag}(\tau_k)$ is self-adjoint on $\mathscr{H}^1$, $U_1(t) = e^{itT}$ is a strongly continuous unitary group extending $e^{-t/2}e^{t\Theta_1}$, its generator iT is skew-adjoint, and

V^1 is a core for it, as in the proof of Proposition 7.46(iii). In degrees 0 and 2, $U_j(t) = 1$ on a line. This is AW5. AW6′ follows from Theorem 7.43 and (i). □

Definition 7.48. Let $g \in C_c^\infty(\mathbb{R})$, $L > 0$, $q > 1$, and let $N \geq 2$ be an integer.

(a) The *reflection* of g is the function $g^\dagger(t) = e^{-t}\overline{g(-t)}$, and the transform of g is $G(s) = \int_{\mathbb{R}} g(t)e^{st}\,dt$.
(b) The *half-shifted coset* is $C_L = \{\frac{1}{2} + \frac{2\pi i}{L}(n + \frac{1}{2}) : n \in \mathbb{Z}\}$.
(c) The *phantom set* is $S_N = \{\sqrt{q}\, e^{i\pi(2m+1)/N} : 0 \leq m < N\}$.

Proposition 7.49 (Weil; [Bom06, §V, p. 9])**.** *Let K and the datum be as in Theorem 7.47, and suppose it satisfies AW6′. Let $g \in C_c^\infty(\mathbb{R})$, with reflection $g^\dagger$ and transform G (Definition 7.48). Then the transform of $g * g^\dagger$ is $G(s)\overline{G(1-\bar{s})}$, and*

$$G_K(g * g^\dagger) = 2\,\mathrm{Re}\big(G(0)\overline{G(1)}\big) - \sum_\rho \mathrm{ord}_\rho(\xi_K)|G(\rho)|^2 \leq 2\,\mathrm{Re}\big(G(0)\overline{G(1)}\big).$$

In particular $G_K(g * g^\dagger) \leq 0$ when $G(0) = G(1) = 0$.

Proof. Substituting $u = -t$, $\int g^\dagger(t)e^{st}dt = \overline{\int g(u)e^{(1-\bar{s})u}du} = \overline{G(1-\bar{s})}$, and convolution becomes multiplication. By Theorem 7.47(ii), every ρ in $\mathrm{Sp}_1 = \mathrm{div}\,\xi_K$ satisfies $1 - \bar{\rho} = \rho$, so the transform at ρ is $|G(\rho)|^2$; at 0 and 1 it is $G(0)\overline{G(1)}$ and $G(1)\overline{G(0)}$. Insert these into (AW6′). □

For $K = \mathbb{Q}$ and $G(0) = G(1) = 0$ this is Weil's positivity criterion in the form of Theorem 11.28 and Proposition 12.7. The terms from degrees 0 and 2 enter the alternating trace with sign + and are not squares; the two linear conditions $G(0) = G(1) = 0$ remove them, which is the role of the poles 0 and 1 in Weil's criterion.

The remaining statements concern test classes that do not determine the spectrum. It is natural to expect that the trace formula for test functions supported in a bounded interval $(0, L)$, which involves the prime ideals with $N\mathfrak{p} < e^L$ and the archimedean places, determines the degree-one spectrum, since such test functions separate the points of the spectrum; in fact it does not.

Theorem 7.50. *Let K be a number field, $L > 0$, and C_L the half-shifted coset (Definition 7.48).*

(i) For every $\varphi \in C_c^\infty(\mathbb{R})$, $\sum_{\rho \in C_L} \Phi(\rho) = L\sum_{k\in\mathbb{Z}}(-1)^k e^{kL/2}\varphi(kL)$.
(ii) Let $\Sigma = \mathrm{div}\,\xi_K + \sum_{\rho\in C_L}[\rho]$. The trivial model with spectrum Σ satisfies AW1 for $(\Lambda_K, \nVdash)$, AW3, (G), and AW6′ on $\mathcal{S}_{(0,L)}$; its degree-one spectrum is $\Sigma \neq \mathrm{div}\,\xi_K$, so it does not satisfy AW2.
(iii) If every zero of ξ_K has real part $\frac{1}{2}$, that trivial model also has positive definite star forms and satisfies AW4 and AW5.

Proof of (i). With $\psi(t) = \varphi(t)e^{t/2}e^{i\pi t/L}$, the Poisson summation formula gives $\sum_n \Phi(\frac{1}{2} + \frac{2\pi i}{L}(n + \frac{1}{2})) = \sum_n \int \psi(t)e^{2\pi i n t/L}dt = L\sum_k \psi(kL)$, and $\psi(kL) = (-1)^k e^{kL/2}\varphi(kL)$. □

Proof of (ii). The set C_L is discrete, invariant under $\rho \mapsto 1 - \rho$ ($n \mapsto -n - 1$), and does not contain $\frac{1}{2}$; so Σ satisfies the hypotheses of Definition 7.5, the multiplicity of $\frac{1}{2}$ being $\mathrm{ord}_{1/2}\,\xi_K$, which is even (proof of Proposition 7.7(i)). Proposition 7.6(i) gives AW1, AW3, and $\mathrm{Sp}_1 = \Sigma$; (G) holds because Σ lies in $0 \leq \mathrm{Re}\,s \leq 1$ with counting function $O(R\log R)$. For $\varphi \in \mathcal{S}_{(0,L)}$, by Theorem 7.41 and (i) the left side of (AW6′) is $G_K(\varphi) - L\sum_k(-1)^k e^{kL/2}\varphi(kL) = G_K(\varphi)$, since no point kL lies in $(0, L)$. □

Proof of (iii). Σ then lies on $\mathrm{Re}\,s = \frac{1}{2}$. Apply Proposition 7.6(ii), and the proof of Theorem 7.47(iii), which uses only that the spectrum lies on the line. □

It is natural to expect that a class of test functions determines the spectrum as soon as it separates the points of the spectrum; in fact separation is necessary but not sufficient. The window class $\mathcal{S}_{(0,L)}$ separates points, since $e^{\rho t} \neq e^{\rho' t}$ on every interval when $\rho \neq \rho'$, and still fails (Theorem 7.50), while every tail class succeeds (Proposition 7.42). The failure is a Poisson-summation phenomenon of infinite spectra: an infinite exponential sum can vanish on an interval. The arithmetic form of "all powers of Frobenius" is therefore a class of test functions whose supports reach arbitrarily far, not a class large enough to separate eigenvalues. The coset C_L is the logarithmic suspension of phantom Frobenius eigenvalues, as the next proposition shows.

It is natural to expect that finitely many Lefschetz identities determine the characteristic polynomial of Frobenius; in fact they do so only together with a bound on the dimension.

Proposition 7.51. *Let $q > 1$, let $N \geq 2$ be an integer, and let S_N be the phantom set (Definition 7.48).*

(i) $\sum_{\alpha \in S_N} \alpha^k = 0$ for $1 \leq k < N$ and $\sum_{\alpha \in S_N} \alpha^N = -Nq^{N/2}$; every $\alpha \in S_N$ has $|\alpha| = \sqrt{q}$, and S_N is stable under $\alpha \mapsto q/\alpha = \bar{\alpha}$.

(ii) The logarithmic suspension of $(\mathbb{C}^{S_N}, \operatorname{diag}(S_N))$ has spectrum $C_{N \log q}$.

Proof of (i). $\sum_m e^{i\pi(2m+1)k/N} = e^{i\pi k/N} \sum_{m=0}^{N-1} e^{2\pi i m k/N}$ vanishes unless $N \mid k$, and equals $Ne^{i\pi}$ for $k = N$. The modulus is clear, and $q/\alpha = \sqrt{q}\, e^{-i\pi(2m+1)/N} = \sqrt{q}\, e^{i\pi(2(N-1-m)+1)/N}$. □

Proof of (ii). With $L = N \log q$, the progression $\Sigma_q(\alpha)$ of $\alpha = \sqrt{q}\, e^{i\pi(2m+1)/N}$ is $\{\frac{1}{2} + \frac{2\pi i}{L}(m + \frac{1}{2} + nN) : n \in \mathbb{Z}\}$, and as m runs over $0, \dots, N-1$ and n over $\mathbb{Z}$, $m + nN$ runs over $\mathbb{Z}$ once. □

So for a curve C over $\mathbb{F}_q$, adding $\operatorname{diag}(S_N)$ to F on V^1 leaves the numbers N_k of Theorem 7.45 for $1 \leq k < N$ unchanged, preserves purity and the symmetry $\alpha \mapsto q/\alpha$, and changes the characteristic polynomial on V^1. For a fixed curve the Newton identities recover that polynomial from $N_1, \dots, N_{2g}$ once the genus g is known; the arithmetic spectrum is infinite, and no window suffices.

It is natural to expect that the two-sided window $(-L, L)$ gives no more information than $(0, L)$, by the symmetry $\rho \mapsto 1 - \rho$; in fact the atom of the geometric side at $t = 0$ detects every spectrum added on the critical line. For the datum of Theorem 7.50, (i) shows that its spectral side differs from $G_K(\varphi)$ by $L\varphi(0)$ on $\mathcal{S}_{(-L,L)}$.

Proposition 7.52. *Let K be a number field and let a Weil datum of type $(1, 0)$ on $\overline{X}_K$ satisfy AW1 for $(\Lambda_K, \nVdash)$, with (V^1, Θ_1) satisfying (G), $\operatorname{spec} \Theta_1 \subset \{\operatorname{Re} s = \frac{1}{2}\}$, and $m_1(\rho) \geq \operatorname{ord}_\rho \xi_K$ for every ρ. If the datum satisfies AW6′ on $\mathcal{S}_{(-L,L)}$ for some $L > 0$, then $\operatorname{Sp}_1 = \operatorname{div} \xi_K$.*

Proof. Put $c = m_1 - \operatorname{ord}(\xi_K) \geq 0$; it is supported on the line. Subtracting Theorem 7.41, $\sum_\rho c_\rho \Phi(\rho) = 0$ for $\varphi \in \mathcal{S}_{(-L,L)}$. For $g \in C_c^\infty((-L/2, L/2))$ the function $g * g^\dagger$ of Proposition 7.49 lies in $\mathcal{S}_{(-L,L)}$, and its transform at a point ρ of the line is $|G(\rho)|^2$, because $1 - \bar{\rho} = \rho$. So $\sum_\rho c_\rho |G(\rho)|^2 = 0$, and $c_\rho G(\rho) = 0$ for all ρ and g. For $\rho = \frac{1}{2} + i\gamma$ take $g \geq 0$, not identically zero, supported in an interval about 0 so short that $\operatorname{Re}(e^{t/2} e^{i\gamma t}) > 0$ on it; then $\operatorname{Re} G(\rho) > 0$. So $c = 0$. □

The hypotheses of Proposition 7.52 put the whole spectrum, hence every zero of ξ_K, on the critical line; the proposition is a uniqueness statement under that assumption. Whether a spectrum that omits some zeros of ξ_K can satisfy AW6′ on a two-sided window is open (Question 13.3).

It is natural to expect that any Schwartz function f of the spectral variable τ can serve as a test function, via "$\operatorname{tr} f(T_1)$"; in fact the prime sum of the geometric side then diverges in general.

Proposition 7.53. *Let K be a number field, $g(t) = \exp(-(1+t^2)^{1/4})$, and $\varphi(t) = e^{-t/2}g(t)$. Then g is a Schwartz function, so $f(\tau) = \int_{\mathbb{R}} \varphi(t)e^{(1/2+i\tau)t}\,dt = \int_{\mathbb{R}} g(t)e^{i\tau t}\,dt$ is a Schwartz function of τ; but $\sum_{\mathfrak{p}} \log N\mathfrak{p}\,\varphi(\log N\mathfrak{p})$ diverges.*

Proof. Every derivative of g is g times a polynomial in derivatives of $(1+t^2)^{1/4}$, which are bounded, and g decays faster than every power of $|t|$; so g is a Schwartz function and so is its Fourier transform f. For $t \geq 1$, $(1+t^2)^{1/4} \leq 2^{1/4}t^{1/2}$, so $\varphi(\log x) \geq x^{-1/2}\exp(-2^{1/4}(\log x)^{1/2}) \geq x^{-3/4}$ for x large. Hence the terms of the sum are eventually at least $N\mathfrak{p}^{-1}$. Since ζ_K has a pole at $s=1$ (Theorem 3.5), $\log\zeta_K(\sigma) = \sum_{\mathfrak{p}}\sum_k N\mathfrak{p}^{-k\sigma}/k$ tends to infinity as $\sigma \to 1^+$, while the terms with $k \geq 2$ stay bounded; so $\sum_{\mathfrak{p}} N\mathfrak{p}^{-1} = \infty$. □

The admissible test functions are transforms of φ for which the prime sum converges and the spectral side can be evaluated at complex ρ; $C_c^\infty(\mathbb{R})$ is the natural class, used throughout.

8. The Künneth conjecture

Weil's proof of the Riemann hypothesis for a curve C over $\mathbb{F}_q$ takes place on the surface $C \times C$: the Frobenius graph and the diagonal are correspondences, and the positivity it uses is Castelnuovo's positivity of the trace form on correspondences, a consequence of the Hodge index theorem on $C \times C$ (Section 4.3). A Weil cohomology over a field has a Künneth formula, which is what places these classes in the cohomology of the surface.

This section asks what the analogous statement for $\overline{X}_K$ should be. It first fixes the algebra of tensor products of flows and the data a cohomology of the "arithmetic surface" $\overline{X}_K \boxtimes \overline{X}_L$ would carry (Section 8.1); it then records the refutations, which show that the natural strong forms of a Künneth formula are inconsistent with the axioms of Section 5 (Section 8.2); it states the conjecture in the form that survives, with the two notions of compactness separated (Section 8.3); and it proves what holds: a Künneth formula relative to the periodic unit over $\mathbb{F}_q$ and for Tate objects and an Eilenberg–Moore formula over a base (Section 8.4), and a Hodge index statement on the arithmetic surface which is equivalent to purity together with definiteness of the star form, the exact counterpart of the positivity in Weil's proof (Section 8.5). No surface is constructed. A Künneth formula for the cohomology of $\operatorname{Spec}\mathbb{Z}\boxtimes\operatorname{Spec}\mathbb{Z}$, with flow $\Theta\otimes 1+1\otimes\Theta$, is already contemplated by Deninger, who observes that the resulting regularized product over the sums $\rho+\rho'$ of zeros does not make sense and relates this to Kurokawa's restriction to zeros with imaginary parts of equal sign [Den94, (7.32), pp. 41–42].

8.1. Flow modules and surface data.

Definition 8.1. Let $q > 1$ be real.

(a) A *flow module* is a complex vector space V with a linear map $\Theta_V : V \to V$; a morphism of flow modules is a linear map commuting with the maps Θ.
(b) The *tensor product* of flow modules (V, Θ_V) and (W, Θ_W) is $(V \otimes_{\mathbb{C}} W, \Theta_V \otimes 1 + 1 \otimes \Theta_W)$.
(c) A *flow algebra* is a commutative unital $\mathbb{C}$-algebra R with a derivation Θ_R; a *flow R-module* is an R-module M with a linear map Θ_M such that $\Theta_M(rm) = \Theta_R(r)m + r\Theta_M(m)$.
(d) The *relative tensor product* $M \otimes_R N$ of flow R-modules is the quotient of $M \otimes_{\mathbb{C}} N$ by the span of the elements $rm \otimes n - m \otimes rn$, with the map induced by $\Theta_M \otimes 1 + 1 \otimes \Theta_N$.
(e) The *periodicity action* on the logarithmic suspension $(\widehat{V}, \Theta)$ of an invertible linear map on a finite-dimensional space (Definition 4.15) is the action of the q-periodic unit $\mathbb{1}_q = \mathbb{C}[v^{\pm 1}]$ (Definition 1.3) in which v sends the copy of u in $\widehat{V}_\rho$ to the copy of u in $\widehat{V}_{\rho+2\pi i/\log q}$.

The rule $\Theta\otimes 1+1\otimes\Theta$ is the one Deninger derives from a hypothetical Künneth formula [Den94, (7.32)]; compare Manin's additive tensor product [Man95, (1.22)].

The map in (d) is well defined: $\Theta_M\otimes 1+1\otimes\Theta_N$ sends $rm\otimes n-m\otimes rn$ to the sum of the three relations $(\Theta_R r)m\otimes n-m\otimes(\Theta_R r)n$, $r(\Theta_M m)\otimes n-\Theta_M m\otimes rn$, and $rm\otimes\Theta_N n-m\otimes r\Theta_N n$. In (e), $\mathbb{K}_q$ is a flow algebra ($\Theta v^n=(2\pi i n/\log q)v^n$ is a derivation), the copy of $u\in V_{(q^\rho)}$ has the same generalized eigenspace in both summands, and $\Theta(v\cdot x)=v\cdot\Theta x+(2\pi i/\log q)v\cdot x$, so $\widehat{V}$ is a flow $\mathbb{K}_q$-module.

Proposition 8.2. *Let (V,Θ_V) and (W,Θ_W) be spectral cores (Definition 1.1) with multiplicities m_V, m_W. Then $V\otimes W$ is the direct sum of the finite-dimensional Θ-stable subspaces $V_{\Theta\sim\rho}\otimes W_{\Theta\sim\rho'}$, on which $\Theta=\Theta_V\otimes 1+1\otimes\Theta_W$ has the single eigenvalue $\rho+\rho'$; for every $\lambda\in\mathbb{C}$,*

$$(V\otimes W)_{\Theta\sim\lambda}=\bigoplus_{\rho+\rho'=\lambda}V_{\Theta\sim\rho}\otimes W_{\Theta\sim\rho'},\qquad \dim(V\otimes W)_{\Theta\sim\lambda}=\sum_{\rho+\rho'=\lambda}m_V(\rho)m_W(\rho').$$

Proof. On $V_{\Theta\sim\rho}\otimes W_{\Theta\sim\rho'}$ the commuting maps $(\Theta_V-\rho)\otimes 1$ and $1\otimes(\Theta_W-\rho')$ are nilpotent, so $\Theta-\rho-\rho'$ is nilpotent there, and on the blocks with $\rho+\rho'\neq\lambda$ the map $\Theta-\lambda$ is invertible. A vector is in the generalized λ-eigenspace if and only if each of its block components is. □

The sums may be infinite; this is the phenomenon behind Theorem 8.12.

Proposition 8.3. *Let $(V^\bullet,\Theta,\cup,\mathrm{Tr},*)$ be a Weil datum of type $(d,0)$ satisfying the derivation rule of AW1 and the trace equivariance of AW3. Then B_j vanishes on $V^j_{\Theta\sim\rho}\times V^{2d-j}_{\Theta\sim\sigma}$ unless $\rho+\sigma=d$.*

Proof. By the derivation rule and the trace equivariance, $B_j(\Theta u,v)+B_j(u,\Theta v)=dB_j(u,v)$. So the linear form on $V^j_{\Theta\sim\rho}\otimes V^{2d-j}_{\Theta\sim\sigma}$ induced by B_j vanishes on the image of $\Theta\otimes 1+1\otimes\Theta-d$. That map has the single eigenvalue $\rho+\sigma-d$ on this finite-dimensional space (proof of Proposition 8.2), so it is surjective when $\rho+\sigma\neq d$. □

Definition 1.37 is stated in the introduction.

The trivial model of Definition 7.5 with spectrum on the line satisfies it: for a pair (e,f), $B_1(*_1(ae+bf),*_1(a'e+b'f))=\bar{a}\bar{b}'-\bar{b}\bar{a}'$. Positivity of h_1 is not part of the condition.

Definition 1.38 is stated in the introduction.

The Hodge index clause is the positivity of the star form h^2_S on N_S, that is, the Castelnuovo (Hodge–Riemann) form of the index theorem on the classes coming from $V^1_K\otimes V^1_L$; it is not a statement about the signature of the duality pairing $(u,v)\mapsto\mathrm{Tr}_S(u\cup v)$ on H^2, which is the arithmetic counterpart of the intersection pairing. The notation $\overline{X}_K\boxtimes\overline{X}_L$ for the object whose cohomology a surface datum would be is only notation: no such object is defined here. The algebraic part plays the role of the classes of correspondences: by Proposition 8.2, N_S lies in the generalized eigenspace of Θ_S at 1, the analogue of the eigenvalue q of Frobenius on the algebraic classes in $H^2(C\times C)$.

Definition 1.39 is stated in the introduction.

The Tate datum is the cohomology of $\mathbb{P}^n$ with its Tate twists; its spectrum in degree $2i$ is $\{i\}$. When $K=L$ and the spectra of both factors lie on the line, the following subspace is the one singled out by Kurokawa's splitting of the zeros into the halves $\mathrm{Im}\,\rho>0$ and $\mathrm{Im}\,\rho<0$, built into his definition of tensor products of zeta functions [Kur92, p. 219] and proposed to define the tensor square of ζ [Man95, §5.2, p. 158].

Definition 8.4. Let (V,Θ) be a spectral core. The *Kurokawa cone* of $V\otimes V$ is the sum of the subspaces $V_{\Theta\sim\rho}\otimes V_{\Theta\sim\rho'}$ over the pairs with $\mathrm{Im}\,\rho\cdot\mathrm{Im}\,\rho'\geq 0$.

8.2. **Refutations.** The findings of this subsection are stated first, before the positive results they delimit.

It is natural to expect a Künneth formula for fibre products over $\operatorname{Spec}\mathbb{Z}$, as for products over a field; in fact the scheme-theoretic square of $\operatorname{Spec}\mathbb{Z}$ over itself is $\operatorname{Spec}\mathbb{Z}$, and a datum is never its own tensor square.

Proposition 8.5. *Let a Weil datum $(V^\bullet, \Theta, \cup, \mathrm{Tr}, *)$ of type $(1,0)$ on $\overline{X}_{\mathbb{Q}}$ satisfy AW1 for $U = \nVdash$ and AW2 for $(\Lambda, \nVdash)$. Then there is no isomorphism of flow modules between V^1 and the degree-one part $V^0 \otimes V^1 \oplus V^1 \otimes V^0$ of its tensor square.*

Proof. By AW1, $V^0 = \mathbb{C}$ with $\Theta_0 = 0$, so the degree-one part of the tensor square is isomorphic to $V^1 \oplus V^1$, with multiplicities $2m_1(\rho)$. By the proof of Theorem 6.3(i), which uses only AW1 and AW2, $\mathrm{Sp}_1 = \operatorname{div}\xi_{\mathbb{Q}}$, and ξ has a zero (proof of Proposition 7.27); at such a zero $2m_1(\rho) \neq m_1(\rho)$. □

It is natural to expect at least a derived Künneth formula over a base, the cohomology of $X \times_S Y$ being that of the derived tensor product of the cohomologies of X and Y over that of S; in fact this fails already in étale cohomology of curves over $\mathbb{P}^1$.

Proposition 8.6. *Let E be an elliptic curve over $\mathbb{F}_q$, $x : E \to \mathbb{P}^1_{\mathbb{F}_q}$ a finite morphism of degree two, $\ell \nmid q$ a prime, $R = H^\bullet(\mathbb{P}^1_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$, and $M = H^\bullet(E_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$, a graded R-module through x^*.*

(i) $R = \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell h$ with h of degree two and $h^2 = 0$; $H^1(E_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ is a two-dimensional R-submodule of M killed by h, and it is a direct summand of M.
(ii) The graded module $\operatorname{Tor}_1^R(M, M)$ has a nonzero component in internal degree four.
(iii) $H^j((E \times_{\mathbb{P}^1} E)_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) = 0$ for $j \geq 3$.

Proof of (i). By Theorem 4.9 for $\mathbb{P}^1$, $H^0 = \mathbb{Q}_\ell$ and H^2 is a line on which Frobenius is q. Since $\#\mathbb{P}^1(\mathbb{F}_{q^n}) = q^n + 1$, $Z(\mathbb{P}^1, t) = 1/((1-t)(1-qt))$, so by Theorem 3.8 the characteristic polynomial of Frobenius on H^1 is 1; Frobenius is injective (remark after Theorem 4.9), so $H^1 = 0$. The product h^2 lies in $H^4 = 0$. The cohomology of E vanishes in degrees above two and $H^1(E)$ has dimension two (the facts recalled after Theorem 3.8), so $h \cdot H^1(E) \subset H^3(E) = 0$, and $M = (H^0 \oplus H^2) \oplus H^1$ as R-modules. □

Proof of (ii). By (i), $\operatorname{Tor}_1^R(M, M)$ contains $\operatorname{Tor}_1^R(H^1, H^1)$ as a direct summand, and H^1 is the sum of two copies of R/hR placed in degree one. The complex $R[-2] \xrightarrow{h} R \to R/hR \to 0$, with $R[-2]$ the module R with degrees raised by two, extends to a graded free resolution of R/hR, because the kernel of multiplication by h on R is hR. After tensoring with R/hR the map h becomes zero, so $\operatorname{Tor}_1^R(R/hR, R/hR)$ is a copy of R/hR in degree two, and $\operatorname{Tor}_1^R(H^1, H^1)$ is four copies of it in degree $1 + 1 + 2 = 4$. □

Proof of (iii). The fibre product is finite over E, hence of dimension one; its reduced subscheme is a reduced separated scheme of finite type and of dimension one, in general reducible (a variety in the sense of the source), which has the same étale site [Mil13, pp. 22–23], and a variety of dimension one over an algebraically closed field has cohomological dimension at most two [Mil13, Theorem 15.1, p. 105]. Passing to the limit over $\mathbb{Z}/\ell^n$ and tensoring with $\mathbb{Q}_\ell$ gives the claim. □

In the derived tensor product $M \otimes^{\mathbf{L}}_R M$, the component of $\operatorname{Tor}_1^R(M, M)$ of internal degree four sits in total degree $4 - 1 = 3$; so the cohomology of $E \times_{\mathbb{P}^1} E$ is not that of the derived tensor product over R.

It is natural to expect that the derived tensor product over the cohomology ring fails only because it forgets the cochains, and that over the étale cochain algebras the formula for $E \times_{\mathbb{P}^1} E$

is repaired; in fact the derived tensor product of the cochain algebras has cohomology of the same dimensions as the one over R, nonzero in every degree.

Definition 8.7. Let k be a field.

(a) For a differential graded k-algebra A, cohomologically graded, a right differential graded A-module B, and a left one B', the *derived tensor product* $B \otimes^{\mathbf{L}}_A B'$ is the complex $B \otimes_A \mathrm{total}(P^\bullet)$ for a proper projective resolution $P^\bullet$ of B' [McC01, pp. 238–239]; its cohomology is the differential $\mathrm{Tor}^A(B, B')$ [McC01, Definition 7.5, p. 240]. A graded algebra is regarded as a differential graded algebra with zero differential.

(b) Let E, x, and ℓ be as in Proposition 8.6, with $\ell \neq 2$, and let k be $\mathbb{F}_\ell$ or $\mathbb{Q}_\ell$. A *dg model of x^* over k* is a homomorphism $A \to B$ of differential graded k-algebras together with isomorphisms of graded algebras $H^\bullet(A) \cong H^\bullet(\mathbb{P}^1_{\overline{\mathbb{F}}_q}, k)$ and $H^\bullet(B) \cong H^\bullet(E_{\overline{\mathbb{F}}_q}, k)$ that carry the map induced on cohomology to x^*; B is a right and a left A-module through $A \to B$.

The étale cochain algebras $C^\bullet(\mathbb{P}^1_{\overline{\mathbb{F}}_q}, k) \to C^\bullet(E_{\overline{\mathbb{F}}_q}, k)$, with the pullback along x, are meant as dg models in this sense; the statements below hold for every dg model.

Proposition 8.8. *Let k be a field, $A \to B$ a homomorphism of differential graded k-algebras, $R = H^\bullet(A)$, and $M = H^\bullet(B)$. Suppose that $R = k \oplus kh$ with h of degree two, that $M^j = 0$ for $j \notin \{0, 1, 2\}$, that $M^0 = k \cdot 1$, that $\dim M^1 = 2$, and that M^2 is one-dimensional and spanned by the image of h. Then $H^n(B \otimes^{\mathbf{L}}_A B)$ has dimension 1, 4, 5 for $n = 0, 1, 2$ and dimension 4 for every $n \geq 3$.*

Proof. The image of h in M^2 is $h \cdot 1 = 1 \cdot h$, and h kills M^1 and M^2 from either side, since $M^3 = M^4 = 0$; so the left and right R-module structures on M agree, the map $R \to M$, $r \mapsto r \cdot 1$, is injective with image $M^0 \oplus M^2$, and $M = R \oplus M^1$ with M^1 the sum of two copies of R/hR placed in degree one. As in the proof of Proposition 8.6(ii), the kernel of multiplication by h on R is hR, so $\cdots \xrightarrow{h} R[-4] \xrightarrow{h} R[-2] \xrightarrow{h} R \to R/hR \to 0$ is a graded free resolution, and $\mathrm{Tor}^R_i(R/hR, R/hR)$ is one-dimensional, in internal degree $2i$, for every $i \geq 0$. Since Tor^R_i vanishes for $i \geq 1$ when one argument is R, the module $\mathrm{Tor}^R_0(M, M) = M \otimes_R M$ has dimensions $1, 4, 5$ in internal degrees $0, 1, 2$ and vanishes in other degrees, and for $i \geq 1$ the module $\mathrm{Tor}^R_i(M, M) = \mathrm{Tor}^R_i(M^1, M^1)$ is four-dimensional and concentrated in internal degree $2 + 2i$.

The filtration of $B \otimes_A \mathrm{total}(P^\bullet)$ by the resolution degree, $\{0\} = F^1 \subset F^0 \subset F^{-1} \subset \cdots$, is exhaustive, and its spectral sequence has $E_2^{-i,j} = \mathrm{Tor}^R_i(M, M)^j$ (internal degree j) [McC01, Theorem 7.6, p. 241] and differentials $d_r : E_r^{-i,j} \to E_r^{-i+r, j-r+1}$ [McC01, Theorem 2.6, p. 33]. Since $F^1 = \{0\}$ the filtration is weakly convergent [McC01, p. 62], so $E_\infty^{-i,j}$ is the graded piece of an exhaustive filtration of $H^{j-i}(B \otimes^{\mathbf{L}}_A B)$ [McC01, Theorem 3.2, p. 63]. The E_2-term is nonzero only at $(0, 0)$, $(0, 1)$, $(0, 2)$, and $(-i, 2 + 2i)$, $i \geq 1$. For $r \geq 2$, d_r leaves the column $i = 0$ for a column $-i + r > 0$, where E_r is zero; from $(-i, 2 + 2i)$ it reaches $(-(i - r), 3 + 2i - r)$, which for $i = r$ has internal degree $3 + i > 2$ and for $i - r \geq 1$ would require $3 + 2i - r = 2 + 2(i - r)$, that is $r = -1$. So every d_r, $r \geq 2$, vanishes, $E_\infty = E_2$, and $\dim H^n(B \otimes^{\mathbf{L}}_A B) = \sum_i \dim E_2^{-i,n+i}$. For $n \leq 2$ only $i = 0$ contributes, and for $n \geq 3$ only $i = n - 2$. □

Proposition 8.9. *Let E, x, and ℓ be as in Proposition 8.6, with $\ell \neq 2$, let k be $\mathbb{F}_\ell$ or $\mathbb{Q}_\ell$, and let $A \to B$ be a dg model of x^* over k. Then $H^n(B \otimes^{\mathbf{L}}_A B)$ has dimension 1, 4, 5 for $n = 0, 1, 2$ and dimension 4 for every $n \geq 3$, whereas $H^n((E \times_{\mathbb{P}^1} E)_{\overline{\mathbb{F}}_q}, k) = 0$ for $n \geq 3$. In particular $B \otimes^{\mathbf{L}}_A B$ is not quasi-isomorphic to any complex whose cohomology is $H^\bullet((E \times_{\mathbb{P}^1} E)_{\overline{\mathbb{F}}_q}, k)$.*

Proof. We verify the hypotheses of Proposition 8.8. For $k = \mathbb{Q}_\ell$ the dimensions of R and M are those of Proposition 8.6(i). Fix compatible isomorphisms $\mu_{\ell^n} \cong \mathbb{Z}/\ell^n$ over $\overline{\mathbb{F}}_q$. For $k = \mathbb{F}_\ell$, a

complete connected nonsingular curve of genus g over an algebraically closed field has $H^j(\cdot,\mu_\ell)$ of dimensions $1, 2g, 1$ for $j = 0, 1, 2$ and zero for $j > 2$ [Mil13, Proposition 14.2, p. 90]. In both cases $h^2 \in H^4 = 0$, and it remains to show that $x^* : H^2(\mathbb{P}^1_{\overline{\mathbb{F}}_q}, k) \to H^2(E_{\overline{\mathbb{F}}_q}, k)$ is nonzero. For a curve X as above and $n \geq 1$, the Kummer sequence identifies $H^2(X, \mu_{\ell^n})$ with $\mathrm{Pic}(X)/\ell^n$ (proof of [Mil13, Proposition 14.2, p. 90]), and the degree identifies this with $\mathbb{Z}/\ell^n$, because $\mathrm{Pic}^0(X)$ is ℓ-divisible [Mil13, Proposition 14.1, p. 89]. The Kummer sequence is functorial, so x^* becomes the pullback of line bundles, which multiplies degrees by two: the pullback of the divisor of a closed point is its scheme-theoretic fibre, of length $\deg x = 2$ since x is finite and flat. As ℓ is odd, x^* is an isomorphism on $H^2(\cdot, \mu_{\ell^n})$ for every n; take $n = 1$ for $\mathbb{F}_\ell$, and pass to the limit and tensor with $\mathbb{Q}_\ell$ [Mil13, p. 126] for $\mathbb{Q}_\ell$. The vanishing for $n \geq 3$ is Proposition 8.6(iii), whose proof gives it for $\mathbb{F}_\ell$-coefficients before the passage to the limit. □

So the cohomology of the cochain-level derived tensor product has the dimensions of the derived tensor product over R (Proposition 8.8), and the failure of the formula for $E \times_{\mathbb{P}^1} E$ is not a matter of the coefficient algebra.

Remark 8.10. It is natural to expect, from Propositions 8.6 and 8.9, that no derived Künneth formula over a base holds in any form. In fact the failure is one of the map x, not of the base: for every cochain enhancement with $\mathbb{F}_\ell$-coefficients (Definition 8.24) the cochains of $Y \times_S Z$ are the derived tensor product of those of Y and Z over those of S whenever $Rf_*\mathbb{F}_\ell$ lies in the thick subcategory generated by the constant sheaf (Proposition 8.27), for instance for $Y = \mathbb{P}^1 \times F$ over the same base $\mathbb{P}^1$ and every proper $Z \to \mathbb{P}^1$, x included (Proposition 8.29), and x violates this hypothesis (Proposition 8.30). Nor do Propositions 8.5, 8.6, and 8.9, which concern the tensor product over a point and over the cohomology or the cochains of a base, exclude a formula of that kind for the arithmetic surface, which is a product over $\mathrm{Spec}\,\mathbb{F}_1$ and not a fibre product over $\mathrm{Spec}\,\mathbb{Z}$, the square of $\mathrm{Spec}\,\mathbb{Z}$ over itself being $\mathrm{Spec}\,\mathbb{Z}$.

Definition 8.11. Let K be a number field and $D_K = (V_K^\bullet, \Theta, \cup, \mathrm{Tr}_K, *)$ a Weil datum of type $(1,0)$ on $\overline{X}_K$.

(a) $Z(K)$ is the set of zeros of ξ_K.
(b) $V_K = V_K^1$ is the degree-one part, with the pairing $B_K = B_1$ and the star form h_1, and $V_{K,\rho} = V_{K,\Theta\sim\rho}$ for $\rho \in \mathbb{C}$.
(c) The *space of algebraic tensors* is $N_K = \bigoplus_\rho V_{K,\rho} \otimes V_{K,1-\rho} \subset V_K \otimes V_K$.
(d) The *tensor star form* is the sesquilinear form $h_\otimes = h_1 \otimes h_1$ on $V_K \otimes V_K$, $h_\otimes(x\otimes y, x'\otimes y') = h_1(x,x')h_1(y,y')$.
(e) For ρ with $\mathrm{Re}\,\rho = \frac{1}{2}$, the *endomorphism map* is $\Phi_\rho : V_{K,\rho} \otimes V_{K,1-\rho} \to \mathrm{End}(V_{K,\rho})$, $\Phi_\rho(x\otimes y)(z) = B_K(z,y)x$, and when h_1 is nondegenerate on $V_{K,\rho}$, $u \mapsto u^\star$ is the adjoint of $u \in \mathrm{End}(V_{K,\rho})$ with respect to $h_1|_{V_{K,\rho}}$.

The same notation is used for a second field L. When $\mathrm{Re}\,\rho = \frac{1}{2}$ the space $V_{K,1-\rho}$ is $V_{K,\bar\rho}$.

It is natural to expect that the tensor product of the cohomologies of two arithmetic curves is again a spectral core, so that the axioms make sense on the arithmetic surface; in fact duality forces an infinite-dimensional generalized eigenspace, and under a mild hypothesis on the zeros also a finite accumulation point of the spectrum.

Theorem 8.12. *Let K, L be number fields, let D_K and D_L be Weil data of type $(1,0)$ on $\overline{X}_K$ and $\overline{X}_L$ satisfying AW1 for $U = \mathbb{1}\!\!\!/$, AW2, and AW3 for $(\Lambda_K, \mathbb{1}\!\!\!/)$ and $(\Lambda_L, \mathbb{1}\!\!\!/)$, and let S be a surface datum over (D_K, D_L) with $\mathrm{Tr}_S(\mathrm{pr}_1^* x \cup \mathrm{pr}_2^* y) = \mathrm{Tr}_K(x)\,\mathrm{Tr}_L(y)$ for $x \in V_K^2$, $y \in V_L^2$, and use the notation of Definition 8.11.*

(i) κ is injective on $V_K \otimes V_L$, and $\Theta_S\kappa = \kappa(\Theta \otimes 1 + 1 \otimes \Theta)$.

(ii) *The generalized eigenspace of* Θ_S *on* H^2 *at* 1 *contains* $\kappa\big(\bigoplus_{\rho\in Z(K)\cap Z(L)} V_{K,\rho}\otimes V_{L,1-\rho}\big)$, *of dimension* $\sum_{\rho\in Z(K)\cap Z(L)} \operatorname{ord}_\rho \xi_K \operatorname{ord}_\rho \xi_L$, *which is infinite if* $K = L$.

(iii) *If* $K = L$ *and the number of distinct imaginary parts in* $(0, T]$ *of zeros of* ξ_K *is not* $O(T)$ *as* $T \to \infty$, *the set of eigenvalues of* Θ_S *on* H^2 *has a finite accumulation point, with imaginary part* 0 *and real part in* $[0, 2]$.

(iv) *If* $Z(K) \cap Z(L)$ *is infinite, in particular if* $K = L$, *then* (H^2, Θ_S) *is not a spectral core; so no surface datum over* (D_K, D_L) *with multiplicative trace is a Weil datum of type* $(2, 0)$, *and neither AW2 nor AW-det can be imposed on it.*

Proof of (i). Θ_S is a derivation commuting with the pr_i^*, which gives the intertwining. For $x, x' \in V_K$ and $y, y' \in V_L$, associativity and graded commutativity give $\kappa(x\otimes y)\cup\kappa(x'\otimes y') = -\mathrm{pr}_1^*(x\cup x')\cup \mathrm{pr}_2^*(y\cup y')$, so $\mathrm{Tr}_S(\kappa(x\otimes y)\cup\kappa(x'\otimes y')) = -B_K(x,x')B_L(y,y')$, with B_K, B_L the degree-one duality pairings. Let $z = \sum z_{\rho\rho'}$ with $z_{\rho\rho'} \in V_{K,\rho}\otimes V_{L,\rho'}$ and $\kappa(z) = 0$. For $u \in V_{K,1-\rho}\otimes V_{L,1-\rho'}$, Proposition 8.3 gives $0 = \mathrm{Tr}_S(\kappa(z)\cup\kappa(u)) = -(B_K\otimes B_L)(z_{\rho\rho'}, u)$. By AW3, $B_K \otimes B_L$ is a perfect pairing of the finite-dimensional spaces $V_{K,\rho}\otimes V_{L,\rho'}$ and $V_{K,1-\rho}\otimes V_{L,1-\rho'}$, so $z_{\rho\rho'} = 0$. □

Proof of (ii). By the proof of Theorem 6.3(i), the multiplicity of ρ on V_K is $\operatorname{ord}_\rho \xi_K$, and likewise for V_L; and $\operatorname{ord}_{1-\rho}\xi_L = \operatorname{ord}_\rho \xi_L$ by Proposition 3.6. By Proposition 8.2 the blocks $V_{K,\rho}\otimes V_{L,1-\rho}$ lie in the generalized eigenspace of $\Theta\otimes 1 + 1\otimes\Theta$ at 1, and by (i) their images are independent and lie in that of Θ_S. The count of zeros [HSW21, p. 1], of order $(n_K T/\pi)\log T$ up to T with $n_K = [K:\mathbb{Q}]$, shows that $Z(K)$ is infinite. □

Proof of (iii). Let $\gamma_1 < \gamma_2 < \cdots$ be the distinct positive imaginary parts of zeros of ξ_K. If $\gamma_{n+1} - \gamma_n \geq \delta > 0$ for all $n \geq n_0$, at most $n_0 + T/\delta$ of them lie in $(0, T]$; so there are $n_1 < n_2 < \cdots$ with $g_k = \gamma_{n_k+1} - \gamma_{n_k}$ positive and strictly decreasing to 0. Choose zeros ρ_k, ρ_k' with imaginary parts γ_{n_k}, γ_{n_k+1}. Then $1 - \rho_k' \in Z(K)$, and by Proposition 8.2 and (i), $\lambda_k = \rho_k + 1 - \rho_k'$ is an eigenvalue of Θ_S on $\kappa(V_{K,\rho_k}\otimes V_{K,1-\rho_k'}) \subset H^2$. The real parts of zeros lie in $[0,1]$ (Proposition 3.6), so $\operatorname{Re}\lambda_k \in [0,2]$, while $\operatorname{Im}\lambda_k = -g_k$ are distinct and tend to 0. A bounded infinite set of points has an accumulation point. □

Proof of (iv). A spectral core has finite-dimensional generalized eigenspaces, and a spectral cut, hence AW-det, is defined only for spectral cores; AW2 is a condition on a Weil datum, whose components are spectral cores. □

The obstruction of (ii) is the one Manin records for the tensor square of ζ, whose zeros ρ and $1-\rho$ make the eigenvalue 1 of infinite multiplicity [Man95, p. 132, difficulty (i)] and Deninger notes for the Künneth cohomology of $\operatorname{Spec}\mathbb{Z}\boxtimes\operatorname{Spec}\mathbb{Z}$ [Den94, (7.32), p. 42]; here it is forced by the cohomological formalism (pullbacks, cup product, a multiplicative trace, and duality), not by a choice of regularization. The hypothesis of (iii) holds, for instance, if a positive proportion of the zeros are simple, since the count of zeros up to height T is of order $T\log T$; the multiplicity bound $O(\log T)$ for zeros alone only gives a number of distinct imaginary parts at least of order T, which does not exclude $O(T)$, and the accumulation point is not established here without the hypothesis.

It is natural to expect that Kurokawa's cone, which does have finite multiplicities for ζ in the known range, is the cohomology of the arithmetic surface; in fact it is not closed under the cup products that the surface must carry, and it contains none of the algebraic classes attached to nonreal zeros.

Proposition 8.13. *In the situation of Theorem 8.12 with* $K = L$, *let* W_c *be the Kurokawa cone of* $V_K \otimes V_K$ *(Definition 8.4).*

(i) *Every subalgebra of $H^\bullet$ containing $\mathrm{pr}_1^* V_K$ and $\mathrm{pr}_2^* V_K$ contains $\kappa(V_K \otimes V_K)$, and $\kappa(W_c) \neq \kappa(V_K \otimes V_K)$.*
(ii) *$W_c \cap \bigoplus_\rho V_{K,\rho} \otimes V_{K,1-\rho} = \bigoplus_{\rho \in Z(K) \cap \mathbb{R}} V_{K,\rho} \otimes V_{K,1-\rho}$, which is finite-dimensional.*

Proof of (i). Such a subalgebra contains every product $\mathrm{pr}_1^* x \cup \mathrm{pr}_2^* y = \kappa(x \otimes y)$. The function ξ_K is real on $\mathbb{R}$, so $Z(K)$ is stable under conjugation; it is infinite (Theorem 8.12(ii)) and its real points form a discrete subset of $[0,1]$, so it contains some ρ with $\operatorname{Im}\rho > 0$, and $V_{K,\rho} \otimes V_{K,\bar\rho}$ is a nonzero block outside W_c; κ is injective. □

Proof of (ii). $\operatorname{Im}(1-\rho) = -\operatorname{Im}\rho$, so $\operatorname{Im}\rho \cdot \operatorname{Im}(1-\rho) \geq 0$ exactly when ρ is real, and $Z(K) \cap \mathbb{R}$ is finite. □

It is natural to expect that analytic compactness, the compact resolvent of the self-adjoint generators required by AW6, survives the tensor product; in fact it survives for spectra bounded on one side and fails for the symmetric spectra that duality produces.

Definition 8.14. Let $\mathscr{H}_1, \mathscr{H}_2$ be Hilbert spaces with orthonormal bases $(e_i)_{i \in I}$ and $(f_k)_{k \in I'}$, and let $T_1 = \operatorname{diag}(\tau_i)$ and $T_2 = \operatorname{diag}(\tau'_k)$ be real diagonal operators (Definition 6.4).

(a) The *tensor sum* $T_1 \boxplus T_2$ is the diagonal operator $\operatorname{diag}(\tau_i + \tau'_k)$ on the Hilbert tensor product $\mathscr{H}_1 \widehat{\otimes} \mathscr{H}_2$ with orthonormal basis $(e_i \otimes f_k)$.
(b) A self-adjoint operator T has *compact resolvent* if $(T - i)^{-1}$ is compact; its *counting function* $N_T(R)$ is the number of eigenvalues of absolute value at most R, with multiplicity.

Proposition 8.15. *Let T_1, T_2, and $T = T_1 \boxplus T_2$ be as in Definition 8.14.*

(i) *T is self-adjoint and agrees with $T_1 \otimes 1 + 1 \otimes T_2$ on the algebraic tensor product of the spans of the e_i and the f_k.*
(ii) *T has compact resolvent if and only if, for every $R > 0$, only finitely many pairs (i,k) satisfy $|\tau_i + \tau'_k| \leq R$.*
(iii) *If T_1 and T_2 have compact resolvent and $\tau_i \geq -c_1$, $\tau'_k \geq -c_2$ for all i, k, with $c_1, c_2 \geq 0$, then T has compact resolvent and $N_T(R) \leq N_{T_1}(R + c) N_{T_2}(R + c)$ with $c = c_1 + c_2$.*
(iv) *If infinitely many pairs (i,k) satisfy $\tau_i + \tau'_k = 0$, then 0 is an eigenvalue of T of infinite multiplicity and T does not have compact resolvent.*
(v) *Both bounds $N_T(R) \leq N_{T_1}(R + c_2) N_{T_2}(R + c_1)$ with $c_1, c_2 \geq 0$, and $N_T(R) \leq N_{T_1}(R + c) N_{T_2}(R + c)$ with $c = c_1 + c_2$ when some $c_i < 0$, fail in general: with $\{\tau_i\} = \{-10\} \cup \{1, 2, 3, \dots\}$ and $\{\tau'_k\} = \{10, 11, 12, \dots\}$ (so $c_1 = 10$, $c_2 = 0$) the first fails at $R = 0$, and with $\{\tau_i\} = \{10, 11, \dots\}$ and $\{\tau'_k\} = \{0, 1, 2, \dots\}$ (so $c_1 = -10$, $c_2 = 0$) the second fails at $R = 10$.*

Proof of (i). The proof of Proposition 6.5 that a diagonal operator with real entries is self-adjoint applies verbatim, and both operators send $e_i \otimes f_k$ to $(\tau_i + \tau'_k) e_i \otimes f_k$. □

Proof of (ii). $(T - i)^{-1}$ is diagonal with entries $c_{ik} = (\tau_i + \tau'_k - i)^{-1}$. A diagonal operator $\operatorname{diag}(c_n)$ is compact if and only if, for every $\varepsilon > 0$, only finitely many $|c_n|$ are at least ε: if so, it is the norm limit of its finite-rank truncations; if not, infinitely many orthonormal e_n have $\|C e_n - C e_m\| \geq \varepsilon\sqrt{2}$, so their images have no convergent subsequence. Finally $|c_{ik}| \geq \varepsilon$ if and only if $|\tau_i + \tau'_k|^2 \leq \varepsilon^{-2} - 1$. □

Proof of (iii). If $|\tau_i + \tau'_k| \leq R$, then $\tau_i \leq R - \tau'_k \leq R + c_2$ and $\tau_i \geq -c_1$, so $|\tau_i| \leq R + c$ because $c_1, c_2 \geq 0$; likewise $|\tau'_k| \leq R + c$. The argument of (ii), applied to the diagonal operators T_1 and T_2 themselves, shows that the numbers of such i and of such k are at most $N_{T_1}(R + c)$ and $N_{T_2}(R + c)$, both finite; apply (ii) to T. □

Proof of (iv). The vectors $e_i \otimes f_k$ with $\tau_i + \tau'_k = 0$ are orthonormal eigenvectors with eigenvalue 0; apply (ii) with any R. □

Proof of (v). In the first example the pair $(-10, 10)$ gives $N_T(0) \geq 1$, while $N_{T_1}(0) = 0$ because no τ_i vanishes. In the second, the pair $(10, 0)$ gives $N_T(10) \geq 1$, while $R + c = 0$ and $N_{T_1}(0) = 0$. □

Proposition 8.16. *Let K be a number field and let a Weil datum of type $(1, 0)$ on $\overline{X}_K$ satisfy AW1–AW4 for $(\Lambda_K, \Vdash)$. Let T_1 be the diagonal operator of Proposition 6.5 for (V^1, Θ_1, h_1) with $j = 1$. Then 0 is an eigenvalue of infinite multiplicity of $T_1 \boxplus T_1$; in particular $T_1 \boxplus T_1$ has no compact resolvent, and the requirement of AW6 that the spectrum have finite multiplicities fails for it.*

Proof. By Theorem 6.3 every eigenvalue is $\rho = \frac{1}{2} + i\tau$ with $\rho \in Z(K)$ and τ an eigenvalue of T_1. Since ξ_K is real on $\mathbb{R}$ and $\xi_K(s) = \xi_K(1 - s)$, $\bar{\rho} = \frac{1}{2} - i\tau$ is also a zero, so $-\tau$ is an eigenvalue of T_1; there are infinitely many zeros (Theorem 8.12(ii)), hence infinitely many pairs of basis vectors with $\tau_i + \tau_k = 0$. Apply Proposition 8.15(iv). □

On $V^1 \otimes V^1$ the closure of the flow generator is $1 + i(T_1 \boxplus T_1)$, the weight-two normalization of AW5; the eigenvalue 0 of $T_1 \boxplus T_1$ is the eigenvalue 1 of Theorem 8.12(ii). The failure is not a problem of domains (the operator is diagonal) but of symmetry: the pairs $(\rho, 1 - \rho)$ are the shadow of duality on the compact object, and by Proposition 8.15(iii) a spectrum bounded on one side, such as the archimedean tower $\{-2n : n \geq 0\}$ of $\Gamma_{\mathbb{R}}$, would keep a compact resolvent and a polynomial counting function.

It is natural to expect that, after logarithmic suspension over $\mathbb{F}_q$, the suspension of a product is the tensor product of the suspensions; in fact the tensor product over $\mathbb{C}$ has infinite multiplicities, and the correct statement is the tensor product over the periodic unit (Theorem 8.23).

Proposition 8.17. *Let $q > 1$, let V, W be nonzero finite-dimensional complex vector spaces with invertible linear maps F, G, and let $\widehat{V}, \widehat{W}$ be their logarithmic suspensions. Then every generalized eigenspace of $\Theta \otimes 1 + 1 \otimes \Theta$ on $\widehat{V} \otimes_{\mathbb{C}} \widehat{W}$ is either zero or infinite-dimensional.*

Proof. If $\widehat{V}_{\rho_1} \otimes \widehat{W}_{\rho_2}$ is a nonzero block, so is $\widehat{V}_{\rho_1 + 2\pi i k/\log q} \otimes \widehat{W}_{\rho_2 - 2\pi i k/\log q}$ for every $k \in \mathbb{Z}$, and all of these lie in the generalized eigenspace at $\rho_1 + \rho_2$ (Proposition 8.2). □

It is natural to expect the absolute product of two closed orbits, the building blocks of Deninger's picture in which the prime p contributes the periodic unit of period $\log p$ [Den98, pp. 167–168 and p. 181], to be again a well-behaved spectral object; in fact it has infinite multiplicities for one prime, dense spectrum for two, and Kurokawa's cone repairs the first only at the price of unbounded multiplicities.

Proposition 8.18. *Let $p \neq p'$ be primes and write $\Vdash_p$ for the p-periodic unit of Definition 1.3.*

(i) Every eigenvalue $2\pi i k/\log p$ of $\Vdash_p \otimes \Vdash_p$ has infinite multiplicity.

(ii) $\Vdash_p \otimes \Vdash_{p'}$ has the eigenvalues $2\pi i(m/\log p + n/\log p')$, $m, n \in \mathbb{Z}$, each simple, and they are dense in $i\mathbb{R}$.

(iii) On the Kurokawa cone of $\Vdash_p \otimes \Vdash_p$ the eigenvalue $2\pi i k/\log p$ has multiplicity $|k| + 1$.

Proof of (i). $\Vdash_p$ is the suspension of $(\mathbb{C}, 1)$ (Proposition 7.23); apply Proposition 8.17. □

Proof of (ii). If $m/\log p + n/\log p' = m'/\log p + n'/\log p'$, then $(m - m')\log p' = (n' - n)\log p$, that is $p'^{m-m'} = p^{n'-n}$, which by unique factorization forces $m = m'$ and $n = n'$. For the same reason $\theta = \log p/\log p'$ is irrational, so the subgroup $\mathbb{Z} + \theta\mathbb{Z}$ of $\mathbb{R}$ is not cyclic and hence dense, and the eigenvalues are $(2\pi i/\log p)(m + n\theta)$. □

Proof of (iii). The cone is spanned by the $v^m \otimes v^n$ with $mn \geq 0$, and $\#\{(m,n) : m+n = k,\ mn \geq 0\} = |k|+1$. □

It is natural to expect the trace formula of the arithmetic surface to be the product of the explicit formulas of the factors, as a Lefschetz formula summing over isolated closed orbits of the diagonal flow; in fact the closed orbits of the diagonal flow on a product of two closed orbits of lengths $\log p$ and $\log p'$ exist only for $p = p'$ and are never isolated.

Proposition 8.19. *Let $a, b > 0$, let $\mathbb{T} = (\mathbb{R}/a\mathbb{Z}) \times (\mathbb{R}/b\mathbb{Z})$ with the flow $\phi^t(x,y) = (x+t, y+t)$, and let p, p' be primes.*

(i) If $a/b \notin \mathbb{Q}$, no point of $\mathbb{T}$ has a periodic orbit.
(ii) If $a/b \in \mathbb{Q}$, every orbit is closed, and no closed orbit is isolated among closed orbits.
(iii) $\log p / \log p' \in \mathbb{Q}$ if and only if $p = p'$.

Proof of (i) and (ii). $\phi^t(x,y) = (x,y)$ if and only if $t \in a\mathbb{Z} \cap b\mathbb{Z}$, which is nonzero exactly when $a/b \in \mathbb{Q}$, and then does not depend on (x,y). In that case the orbit of (x,y) meets the circle $\{x\} \times \mathbb{R}/b\mathbb{Z}$ in the finite set of points $(x, y+ma)$, $m \in \mathbb{Z}$, so for all small $s \neq 0$ the orbit of $(x, y+s)$ is a different closed orbit, arbitrarily close. □

Proof of (iii). $m \log p = n \log p'$ with positive integers m, n means $p^m = p'^n$. □

A Lefschetz formula that sums over isolated closed orbits therefore does not apply to the product of two closed orbits with the diagonal flow, and for $p \neq p'$ the spectral side of Proposition 8.18(ii) has no geometric counterpart at all. A trace formula on the arithmetic surface, if there is one, must be of the type of non-isolated fixed points.

It is natural to expect the archimedean factor of a tensor product of representations to be obtained from the factors by adding spectra, as the nonarchimedean zeros of Theorem 8.23 are; in fact it depends multiplicatively on the action of complex conjugation, which the flow does not see.

Proposition 8.20. *Let χ be the nontrivial character of $\mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q})$. Then $L_\infty(\chi, s) = \Gamma_{\mathbb{R}}(s+1)$ and $L_\infty(\chi \otimes \chi, s) = \Gamma_{\mathbb{R}}(s)$ (Definition 7.1); the divisor of poles of $\Gamma_{\mathbb{R}}(s)$ is $\sum_{n \geq 0}[-2n]$, whereas the sum of the pole divisor of $\Gamma_{\mathbb{R}}(s+1)$ with itself under $(\rho, \rho') \mapsto \rho + \rho'$ is $\sum_{k \geq 0}(k+1)[-2-2k]$.*

Proof. Complex conjugation acts on $\mathbb{Q}(i)$ nontrivially, so χ restricted to the decomposition group at the real place is the sign character and $\chi \otimes \chi$ is trivial; Definition 7.1(b) gives the two factors. The poles of $\Gamma(s/2)$ are at $s = -2n$ and those of $\Gamma((s+1)/2)$ at $s = -1-2n$, $n \geq 0$, all simple, and $\#\{(m,n) : (-1-2m) + (-1-2n) = -2-2k\} = k+1$. □

The two divisors differ (the first contains 0, the second does not): the datum lost by the flow is the involution at infinity, a multiplicative datum. A relative Künneth formula over the archimedean point, if there is one, must keep that involution before taking invariants.

It is natural to expect that the obstruction to real coefficients is an accident of degree one (Remark 9.21); in fact a Künneth formula turns the coefficients into a tensor functor, and on the finite-field calibration no tensor functor of the relevant kind with real values exists.

Remark 8.21. The underlying obstruction is Serre's (Proposition 9.22). Deninger observes that, for motives over finite fields, exact faithful tensor functors into real vector spaces do not exist [Den98, p. 184]. A cohomology with a Künneth formula, restricted to the finite-field objects, is a tensor functor; so a Künneth-compatible theory which restricts there to an exact faithful tensor functor has non-real coefficients. The quaternionic refinement $*_1^2 = -1$ is compatible with tensor products: on $V^1 \otimes V^1$ the square of $* \otimes *$ is $(-1)(-1) = 1$.

8.3. **The conjecture and compactness.** The refutations leave one form of the conjecture. Two notions of compactness enter it and must be kept apart. *Geometric* compactness is that of $\overline{X}_K$, whose infinite places are included: it is what makes the duality AW3 and the symmetry $\rho \leftrightarrow 1-\rho$ of the spectrum hold, and it is exactly this symmetry that produces the infinite multiplicities of Theorem 8.12. *Analytic* compactness is the compact resolvent of AW6; it survives products of spectra bounded on one side and fails on the tensor square of every compact arithmetic curve (Propositions 8.15 and 8.16). The two requirements are in tension on products. One may either (α) keep the Künneth isomorphism and the algebraic classes, and drop the spectral-core, determinant, and compact-resolvent conditions on the surface, or (β) keep those conditions by passing to a one-sided subspace such as Kurokawa's cone, which is not closed under cup products and contains no algebraic classes with nonreal eigenvalues (Proposition 8.13). Form (β) is refuted; the conjecture is stated in form (α).

Conjecture 1.40 is stated in the introduction.

A Künneth formula for an arithmetic cohomology of $\operatorname{Spec}\mathbb{Z}\boxtimes\operatorname{Spec}\mathbb{Z}$, with flow $\Theta\otimes 1+1\otimes\Theta$, is contemplated in [Den94, (7.32), pp. 41–42], and positivity on a square is the aim of the Riemann–Roch strategy of Connes and Consani [CC19, §3, pp. 16–18]; Conjecture 1.40 is one precise form of these expectations. Section 1.5 explains their absolute-base motivation and distinguishes the constructed sites from the required cohomological and intersection-theoretic comparisons.

The conjecture is meant for data constructed from arithmetic input independent of the zeros of the ξ_K. That requirement is not a mathematical condition, and Proposition 8.37 below shows that without it the conjecture is equivalent to the Riemann hypothesis for every ζ_K, exactly as for the axioms on one object (Proposition 7.7). Its role is the one Grothendieck's construction of $H^\bullet(C\times C)$ plays over $\mathbb{F}_q$: the surface must exist before its positivity is proved. Candidate ambient categories for such a construction include Deninger's dynamical systems, in which the product carries the diagonal flow, and, for the local analogue at p, Scholze's products of diamonds, which "give good meaning to the non-existent products $\operatorname{Spec}\mathbb{Q}_p\times_{\operatorname{Spec}\mathbb{F}_1}\cdots$" and relate a hypothetical $\operatorname{Spec}\mathbb{Z}\times\operatorname{Spec}\mathbb{Z}$ to a geometric realization of the Galois group [Sch18, §6, pp. 13–14]. Relative products are allowed only over unit-like bases: over $\mathbb{F}_q$ the formula is a theorem (Theorem 8.23), over $\operatorname{Spec}\mathbb{Z}$ it is refuted (Proposition 8.5), and over the archimedean point the involution at infinity must be kept (Proposition 8.20).

Table 2: Which compactness each conclusion on the arithmetic surface needs.

Conclusion	Compactness needed	Status
Künneth map injective, intertwining the flows	duality AW3 on the factors (geometric)	Thm. 8.12(i)
algebraic classes N_S and the Hodge index clause	geometric: the classes pair ρ with $1-\rho$	Thm. 8.34
spectral core, determinants, AW2 on the surface	finite multiplicities: contradicted by geometric compactness	refuted, Thm. 8.12(iv)
compact resolvent (AW6) on the surface	analytic; incompatible with the symmetric spectrum	refuted, Prop. 8.16
compact resolvent for one-sided spectra	analytic only	proved, Prop. 8.15(iii)

Geometric compactness enters the divisor identity only through the boundary. For an open object X with compactification $\overline{X}$ and boundary ∂X at the infinite places, one expects compactly supported cohomology $H^\bullet_c(X)$ with a flow and a long exact sequence relating it to $H^\bullet(\overline{X})$ and

$H^\bullet(\partial X)$; Deninger's picture assigns to the infinite place the tower $\{-2n : n \geq 0\}$ of poles of $\Gamma_{\mathbb{R}}$ [Den98, pp. 167–168]. It is natural to expect that the divisor identity for the open object decides the connecting maps of that sequence, for instance whether the eigenvalue 0 of $H^0(\overline{X})$ survives in $H^0_c(X)$; in fact the alternating spectral divisor of any such sequence is independent of the maps.

Proposition 8.22. *Let $0 \to E^0 \to F^0 \to G^0 \to E^1 \to F^1 \to G^1 \to E^2 \to F^2 \to 0$ be an exact sequence of spectral cores and linear maps commuting with the generators. Then*

$$\sum_{j=0}^{2} (-1)^j \operatorname{Sp}(E^j) = \sum_{j=0}^{2} (-1)^j \operatorname{Sp}(F^j) - \sum_{j=0}^{1} (-1)^j \operatorname{Sp}(G^j).$$

In particular, if $F^0 = \mathbb{C}$ with generator 0, and two such sequences have the same spaces F^j and G^j, with $F^0 \to G^0$ injective in the first and zero in the second, then they have the same alternating divisor $\sum_j (-1)^j \operatorname{Sp}(E^j)$, while $\operatorname{Sp}(E^0)$ is 0 in the first case and $[0]$ in the second.

Proof. A linear map commuting with the generators sends each generalized eigenspace into the generalized eigenspace for the same eigenvalue, so the sequence is the direct sum, over $\lambda \in \mathbb{C}$, of its restrictions to the generalized λ-eigenspaces. These are exact sequences of finite-dimensional spaces, so the alternating sum of their dimensions vanishes; this is the displayed identity, coefficient by coefficient. In the special case, E^0 is the kernel of $F^0 \to G^0$. □

So the divisor identity for the compactly supported cohomology of $\operatorname{Spec}\mathbb{Z}$, with its trivial zeros coming from the boundary tower, follows from the identities for $\overline{X}_{\mathbb{Q}}$ and ∂X whatever the connecting maps, and it does not determine whether 0 lies in the spectrum of H^0_c. If moreover the eigenvalue 0 occurs in $H^0(\partial X)$ with multiplicity one, as for the tower $\{-2n : n \geq 0\}$, then, since $\xi(0) \neq 0$ so that 0 does not occur in $H^1(\overline{X}_{\mathbb{Q}})$, the multiplicity of 0 in H^1_c is 1 minus the rank of $H^0(\overline{X}_{\mathbb{Q}}) \to H^0(\partial X)$, and the divisor identity does not determine it either.

8.4. **What holds.** Over a finite field the formula holds once the suspension is tensored over the periodic unit, with determinant normalization exactly one.

Theorem 8.23. *Let $q > 1$, let V, W be finite-dimensional complex vector spaces with invertible linear maps F, G, and let $\widehat{V}$, $\widehat{W}$, and $\widehat{V \otimes W}$ be the logarithmic suspensions of (V, F), (W, G), and $(V \otimes W, \mathrm{F} \otimes \mathrm{G})$, with the periodicity actions of Definition 8.1(e).*

- *(i) There is an isomorphism of flow modules $\widehat{V \otimes W} \to \widehat{V} \otimes_{\mathbb{K}_q} \widehat{W}$ sending the copy of $u \otimes w$, $u \in V_{(\alpha)}$, $w \in W_{(\beta)}$, in the summand indexed by ρ (with $q^\rho = \alpha\beta$) to the class of the copy of u over ρ_1 tensored with the copy of w over $\rho - \rho_1$, for any $\rho_1 \in \Sigma_q(\alpha)$.*
- *(ii) If X, Y are smooth projective geometrically connected varieties over $\mathbb{F}_q$, $\ell \nmid q$, and $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$, then in each degree n the suspended datum of $X \times_{\mathbb{F}_q} Y$ is isomorphic, as a flow module, to $\bigoplus_{a+b=n} \widehat{H}^a_\ell(X) \otimes_{\mathbb{K}_q} \widehat{H}^b_\ell(Y)$, and its principal-cut determinant is $\prod_{a+b=n} \det(1 - \mathrm{F} \otimes \mathrm{G}\, q^{-s} \mid H^a_\ell(X) \otimes H^b_\ell(Y))$, with normalization term 1.*

Proof of (i). Write $c = 2\pi i / \log q$. On $V_{(\alpha)} \otimes W_{(\beta)}$ the commuting maps $\alpha^{-1}\mathrm{F} \otimes 1$ and $1 \otimes \beta^{-1}\mathrm{G}$ are unipotent, so $\mathrm{F} \otimes \mathrm{G}$ has the single eigenvalue $\alpha\beta$ there, and the summand of $\widehat{V \otimes W}$ over ρ is $\bigoplus_{\alpha\beta = q^\rho} V_{(\alpha)} \otimes W_{(\beta)}$. The space $\widehat{V} \otimes_{\mathbb{C}} \widehat{W}$ is the direct sum of the blocks $\widehat{V}_{\rho_1} \otimes \widehat{W}_{\rho_2}$, each a copy of $V_{(q^{\rho_1})} \otimes W_{(q^{\rho_2})}$. The relations $vx \otimes y - x \otimes vy$ identify the block $(\rho_1 + c, \rho_2)$ with the block $(\rho_1, \rho_2 + c)$ through the identity of $V_{(\alpha)} \otimes W_{(\beta)}$, and those for $v^{\pm n}$ follow from those for v (for instance $v^{-1}x \otimes y - x \otimes v^{-1}y$ is minus the relation for v applied to $v^{-1}x \otimes v^{-1}y$). So the quotient is the direct sum, over the orbits of $(kc, -kc)$, $k \in \mathbb{Z}$, on pairs (ρ_1, ρ_2), of one copy of $V_{(\alpha)} \otimes W_{(\beta)}$; and these orbits correspond bijectively to the triples (ρ, α, β) with $\rho = \rho_1 + \rho_2$, $\alpha = q^{\rho_1}$, $\beta = q^{\rho_2}$.

This is the stated map and it is bijective. On $V_{(\alpha)} \otimes W_{(\beta)}$ the logarithm of the unipotent part of $(\alpha\beta)^{-1}\mathrm{F} \otimes \mathrm{G}$ is $N_\alpha \otimes 1 + 1 \otimes N_\beta$ (the logarithm of a product of commuting unipotent maps is the sum of the logarithms), so both generators are $\rho + (N_\alpha \otimes 1 + 1 \otimes N_\beta)/\log q$. □

Proof of (ii). The cup product induces an isomorphism $\bigoplus_{a+b=n} H^a(X) \otimes H^b(Y) \to H^n(X \times Y)$ with $\mathbb{Q}_\ell$-coefficients [Mil13, Theorem 22.4, p. 136] (the source states it with $\mathbb{Z}_\ell$-coefficients and a Tor term, which vanishes after tensoring with $\mathbb{Q}_\ell$). It is compatible with the Frobenius by naturality of the Künneth map in X and Y: the Frobenius of $X \times Y$ is the product of the Frobenius endomorphisms, and pullbacks along it commute with the projections and with cup products. Apply (i) and Theorem 7.24(ii) for $X \times Y$. □

Theorem 8.23 is the spectral shadow of the strong Künneth theorem for topological periodic cyclic homology: for smooth and proper dg categories over a perfect field k of characteristic p, $\mathrm{TP}(X) \wedge^{\mathbf{L}}_{\mathrm{TP}(k)} \mathrm{TP}(Y) \to \mathrm{TP}(X \otimes_k Y)$ is an equivalence [BM17, Theorem A, pp. 2–3], the tensor product being taken over $\mathrm{TP}(k)$ and not over the sphere; Hesselholt identifies the Hasse–Weil zeta function with a regularized determinant on TP [Hes18, Theorem A and footnote 1, p. 2]. The tensor product over $\mathbb{F}_q$ plays the role of the tensor product over $\mathrm{TP}(k)$; over $\mathbb{C}$ it has infinite multiplicities (Proposition 8.17).

Over a base of positive dimension a derived Künneth formula holds under a hypothesis on the map, in the form of an étale Eilenberg–Moore statement, and the double cover of Proposition 8.6 violates that hypothesis.

Definition 8.24. Let K be a separably closed field, ℓ a prime invertible in K, and $\Lambda = \mathbb{F}_\ell$. Schemes are of finite type over K.

(a) For a scheme S, $\langle \Lambda_S \rangle$ is the smallest strictly full triangulated subcategory of $D(S_{\mathrm{et}}, \Lambda)$ that contains the constant sheaf Λ_S and is closed under direct summands.

(b) A *cochain enhancement* assigns to each scheme T a differential graded Λ-algebra $C^\bullet(T)$ whose cohomology algebra is $H^\bullet(T, \Lambda)$ with the cup product, and an exact functor $R\Gamma_T$ from $D^+(T_{\mathrm{et}}, \Lambda)$ to the derived category of differential graded $C^\bullet(T)$-modules which lifts $R\Gamma(T, -)$ and sends Λ_T to $C^\bullet(T)$; to each morphism $f : Y \to S$ a homomorphism $f^* : C^\bullet(S) \to C^\bullet(Y)$ inducing the pullback on cohomology, and natural isomorphisms $R\Gamma_S(Rf_*L) \cong R\Gamma_Y(L)$ of $C^\bullet(S)$-modules, $C^\bullet(Y)$-modules being $C^\bullet(S)$-modules through f^*; and to each S a natural transformation $\mu_S : R\Gamma_S(L) \otimes^{\mathbf{L}}_{C^\bullet(S)} R\Gamma_S(L') \to R\Gamma_S(L \otimes^{\mathbf{L}}_\Lambda L')$ of exact functors in each variable, which for $L = \Lambda_S$ is the canonical identification. Derived tensor products are those of Definition 8.7.

The étale cochain algebras with $\mathbb{F}_\ell$-coefficients of Remark 8.10 are meant to form a cochain enhancement in this sense; Godement resolutions are the usual construction, and their properties are not verified here. The statements below that involve $C^\bullet$ assume a cochain enhancement and hold for every one.

Proposition 8.25. *In the setting of Definition 8.24, let $f : Y \to S$ and $g : Z \to S$ be proper, let $\tilde{g} : Y \times_S Z \to Y$ be the projection and $h = f \circ \tilde{g}$. Then $Rh_*\Lambda \cong Rf_*\Lambda \otimes^{\mathbf{L}}_\Lambda Rg_*\Lambda$ in $D(S_{\mathrm{et}}, \Lambda)$.*

Proof. Let $\tilde{f} : Y \times_S Z \to Z$ be the other projection. By proper base change for g along f [Sta26, Tag 0F0C] 0F0C, $f^{-1}Rg_*\Lambda \cong R\tilde{g}_*\tilde{f}^{-1}\Lambda = R\tilde{g}_*\Lambda$. By the projection formula for the proper morphism f [Sta26, Tag 0F0G] 0F0G, applied to Λ_Y and $Rg_*\Lambda$, $Rf_*\Lambda \otimes^{\mathbf{L}}_\Lambda Rg_*\Lambda \cong Rf_*(f^{-1}Rg_*\Lambda) \cong Rf_*R\tilde{g}_*\Lambda = Rh_*\Lambda$. □

Definition 8.26. In the setting of Proposition 8.25, fix a cochain enhancement. The *Eilenberg–Moore comparison map*

$$\kappa : C^\bullet(Y) \otimes^{\mathbf{L}}_{C^\bullet(S)} C^\bullet(Z) \to C^\bullet(Y \times_S Z)$$

is the composite of the identifications $C^\bullet(Y) \cong R\Gamma_S(Rf_*\Lambda)$ and $C^\bullet(Z) \cong R\Gamma_S(Rg_*\Lambda)$, of μ_S for the pair $(Rf_*\Lambda, Rg_*\Lambda)$, of $R\Gamma_S$ applied to the isomorphism of Proposition 8.25, and of $R\Gamma_S(Rh_*\Lambda) \cong C^\bullet(Y \times_S Z)$.

Proposition 8.27 (after [Lur11, Proposition 1.1.9, p. 6, and Corollary 1.1.10, p. 7])**.** *In the setting of Definition 8.26, if $Rf_*\Lambda$ lies in $\langle \Lambda_S \rangle$, then κ is an isomorphism in the derived category of $C^\bullet(S)$-modules.*

Proof. All maps in the definition of κ other than μ_S are isomorphisms, so it suffices that μ_S is an isomorphism for the pair $(Rf_*\Lambda, Rg_*\Lambda)$. Fix $L' = Rg_*\Lambda$ and let $\mathcal{C}$ be the class of L in $D^+(S_{\mathrm{et}}, \Lambda)$ for which μ_S is an isomorphism at (L, L'). It contains Λ_S. Both sides of μ_S are exact functors of L and μ_S is a natural transformation of exact functors, so a distinguished triangle with two vertices in $\mathcal{C}$ gives a morphism of long exact cohomology sequences which is an isomorphism at two of every three places, and the third vertex lies in $\mathcal{C}$ by the five lemma; $\mathcal{C}$ is closed under shifts, and under direct summands because a retract of an isomorphism is an isomorphism. So $\mathcal{C}$ contains $\langle \Lambda_S \rangle$. □

Proposition 8.28. *In the setting of Definition 8.26, suppose that only finitely many cohomology sheaves $R^i f_*\Lambda$ are nonzero and that each has a finite filtration whose graded pieces are finite direct sums of copies of Λ_S. Then $Rf_*\Lambda$ lies in $\langle \Lambda_S \rangle$, and κ is an isomorphism for every proper $g : Z \to S$.*

Proof. A short exact sequence of sheaves is a distinguished triangle, so each $R^i f_*\Lambda$ lies in $\langle \Lambda_S \rangle$ by induction on the length of the filtration. The truncation triangles $\tau_{\le i-1} Rf_*\Lambda \to \tau_{\le i} Rf_*\Lambda \to R^i f_*\Lambda[-i]$ and induction on the number of nonzero cohomology sheaves give the first claim; apply Proposition 8.27. □

Proposition 8.27 is the étale form of the Eilenberg–Moore theorem in Lurie's formulation, with the same thick-subcategory proof [Lur11, Proposition 1.1.9, p. 6, and Corollary 1.1.10, p. 7]; the category $\langle \Lambda_S \rangle$ corresponds to Lurie's nilpotent objects [Lur11, Definition 1.1.7, p. 6], and the nilpotence condition goes back to Dwyer [Dwy74], as reported in [McC01, Theorem 8^{bis}.30].

Proposition 8.29. *In the setting of Definition 8.26, let K be algebraically closed, F a smooth projective connected curve over K, $Y = S \times_K F$, and f the first projection. Then $Rf_*\Lambda$ lies in $\langle \Lambda_S \rangle$, and for every proper $g : Z \to S$ the map $\kappa : C^\bullet(S \times_K F) \otimes^{\mathbf{L}}_{C^\bullet(S)} C^\bullet(Z) \to C^\bullet(Z \times_K F)$ is an isomorphism.*

Proof. Let $a : S \to \operatorname{Spec} K$ and $b : F \to \operatorname{Spec} K$ be the structure morphisms. By proper base change for b along a [Sta26, Tag 0F0C] 0F0C, $Rf_*\Lambda \cong a^{-1} Rb_*\Lambda$. Since K is separably closed, $Rb_*\Lambda$ is a complex of Λ-vector spaces, hence isomorphic to $\bigoplus_j H^j(F, \Lambda)[-j]$, and $H^j(F, \Lambda)$ has dimension $1, 2g, 1$ for $j = 0, 1, 2$ and vanishes for $j > 2$, with g the genus of F [Mil13, Proposition 14.2, p. 90]. So $Rf_*\Lambda$ is a finite direct sum of shifts of Λ_S, and $Y \times_S Z = Z \times_K F$; apply Proposition 8.28. □

It is natural to expect that the failure of Propositions 8.6 and 8.9 is a defect of the base $\mathbb{P}^1$; in fact over the same base the formula holds for $\mathbb{P}^1 \times F$ and every proper $Z \to \mathbb{P}^1$ (Proposition 8.29), and what fails is the hypothesis of Proposition 8.27 for the map x.

Proposition 8.30. *Let E and x be as in Proposition 8.6, $\ell \nmid 2q$ a prime, $\Lambda = \mathbb{F}_\ell$, and consider x over $K = \overline{\mathbb{F}}_q$. Then $Rx_*\Lambda$ does not lie in $\langle \Lambda_{\mathbb{P}^1} \rangle$, and for every cochain enhancement the map κ for the pair (x, x) is not an isomorphism.*

Proof. Write $S = \mathbb{P}^1_{\overline{\mathbb{F}}_q}$. Let $\mathcal{D}$ be the class of objects of $D(S_{\mathrm{et}}, \Lambda)$ each of whose cohomology sheaves is isomorphic to $\Lambda_S^{\oplus r}$ for some $r \ge 0$. Since S is connected, $\operatorname{Hom}(\Lambda_S^{\oplus a}, \Lambda_S^{\oplus b})$ is the space

of $b \times a$ matrices over Λ, so kernels, cokernels, and images of idempotents of such sheaves are again of this form; and every extension of $\Lambda_S^{\oplus a}$ by $\Lambda_S^{\oplus b}$ splits, because $H^1(S, \Lambda) = 0$ [Mil13, Proposition 14.2, p. 90]. The long exact sequence of cohomology sheaves of a distinguished triangle therefore shows that $\mathcal{D}$ is closed under cones, and it is closed under shifts and direct summands; so $\langle \Lambda_S \rangle \subset \mathcal{D}$. Since x is finite, $R^q x_* \Lambda = 0$ for $q \geq 1$ [Sta26, Tag 03QP] 03QP, so $Rx_*\Lambda = x_*\Lambda$ and $H^1(S, x_*\Lambda) = H^1(E_{\overline{\mathbb{F}}_q}, \Lambda)$, which is two-dimensional [Mil13, Proposition 14.2, p. 90]. If $x_*\Lambda$ were isomorphic to $\Lambda_S^{\oplus r}$, this group would be $H^1(S, \Lambda)^{\oplus r} = 0$. So $Rx_*\Lambda \notin \mathcal{D} \supset \langle \Lambda_S \rangle$.

For the second claim, $x^* : C^\bullet(S) \to C^\bullet(E_{\overline{\mathbb{F}}_q})$ is a dg model of x^* over $\mathbb{F}_\ell$ (Definition 8.7), so by Proposition 8.9 the source of κ has nonzero cohomology in every degree $n \geq 3$, whereas its target has none. □

Over $\mathbb{F}_q$ the tensor product of Theorem 8.23 is taken over the periodic unit, which is the suspension of the cohomology of the base $\operatorname{Spec} \mathbb{F}_q$; over a base with more cohomology the suspended derived tensor product is taken over the suspension of that cohomology.

Proposition 8.31. *Let $q > 1$, let R be a finite-dimensional commutative $\mathbb{C}$-algebra with an algebra automorphism F_R, and let M, N be finite-dimensional R-modules with invertible linear maps F_M, F_N such that $\mathrm{F}_M(rm) = \mathrm{F}_R(r)\mathrm{F}_M(m)$ and $\mathrm{F}_N(rn) = \mathrm{F}_R(r)\mathrm{F}_N(n)$. Let $\widehat{R}$, $\widehat{M}$, $\widehat{N}$ be the logarithmic suspensions.*

(i) Through the isomorphisms of Theorem 8.23(i), the product of R makes $\widehat{R}$ a flow algebra containing $\mathbb{K}_q = \widehat{\mathbb{C}}$, and the actions of R make $\widehat{M}$, $\widehat{N}$ flow $\widehat{R}$-modules.

(ii) For every $i \geq 0$ there is an isomorphism of complex vector spaces $\operatorname{Tor}_i^{\widehat{R}}(\widehat{M}, \widehat{N}) \cong \widehat{\operatorname{Tor}_i^R(M, N)}$, where Tor_i^R carries the automorphism induced by F_M and F_N; for $i = 0$ it is an isomorphism of flow modules $\widehat{M} \otimes_{\widehat{R}} \widehat{N} \cong \widehat{M \otimes_R N}$.

Proof of (i). A linear map commuting with the automorphisms preserves generalized eigenspaces and commutes with the maps N_α of Definition 4.15, so it induces a morphism of flow modules between suspensions. The isomorphisms of Theorem 8.23(i) are, by their explicit formula, natural in both factors and compatible with the associativity, symmetry, and unit isomorphisms, the suspension of $(\mathbb{C}, 1)$ being $\mathbb{K}_q$ (Proposition 7.23): for the symmetry, choose $\rho - \rho_1$ in place of ρ_1 for the exchanged factors; for the unit, the copy of 1 over $\rho - \rho_1$ is $v^k \cdot 1$ with $\rho - \rho_1 = 2\pi i k / \log q$. Composing their inverses with the suspensions of the product $R \otimes R \to R$ and of the actions gives the stated structures; the flow is a derivation because the isomorphisms and the suspended maps are morphisms of flow modules. □

Proof of (ii). Let $\bar{R} = R/\mathbb{C} \cdot 1$ and $P_i = R \otimes \bar{R}^{\otimes i} \otimes M$ with the bar differential; the maps $r[r_1|\cdots|r_i]m \mapsto 1[r|r_1|\cdots|r_i]m$ form a contracting homotopy, so $P_\bullet \to M$ is a resolution by the free modules $R \otimes V_i$, $V_i = \bar{R}^{\otimes i} \otimes M$, and the maps induced by F_R and F_M make it equivariant. Taking generalized eigenspaces is exact, so suspension is exact and $\widehat{P}_\bullet \to \widehat{M}$ is exact. Choosing one $\rho_\alpha \in \Sigma_q(\alpha)$ for each eigenvalue α shows that $\widehat{V}$ is a free $\mathbb{K}_q$-module of rank $\dim V$, so $\widehat{R \otimes V_i} \cong \widehat{R} \otimes_{\mathbb{K}_q} \widehat{V_i}$ is a free $\widehat{R}$-module and $\widehat{P}_\bullet$ is a free resolution of $\widehat{M}$. The maps $\widehat{P} \otimes_{\mathbb{K}_q} \widehat{N} \cong \widehat{P \otimes_{\mathbb{C}} N} \to \widehat{P \otimes_R N}$ are $\widehat{R}$-balanced and natural in P, and for $P = R \otimes V$ they induce the isomorphism $\widehat{P} \otimes_{\widehat{R}} \widehat{N} \cong \widehat{V} \otimes_{\mathbb{K}_q} \widehat{N} \cong \widehat{V \otimes N} = \widehat{P \otimes_R N}$. So $\widehat{P}_\bullet \otimes_{\widehat{R}} \widehat{N} \cong \widehat{P_\bullet \otimes_R N}$ as complexes, and exactness of suspension gives the claim; for $i = 0$ all maps are morphisms of flow modules. □

For $R = \mathbb{C}$ with $\mathrm{F}_R = 1$, the cohomology of the base $\operatorname{Spec} \overline{\mathbb{F}}_q$, one has $\widehat{R} = \mathbb{K}_q$, $\operatorname{Tor}_i^R = 0$ for $i \geq 1$, and the case $i = 0$ of Proposition 8.31(ii) is Theorem 8.23(i); Theorem 8.23(ii) is its application to the cohomology of two varieties over $\operatorname{Spec} \mathbb{F}_q$. For R and M the cohomology of

$\mathbb{P}^1_{\overline{\mathbb{F}}_q}$ and of $E_{\overline{\mathbb{F}}_q}$ in Proposition 8.6, tensored with $\mathbb{C}$ through ι and with their Frobenius maps, $\operatorname{Tor}_i^R(M,M)$ is nonzero for every i (proof of Proposition 8.8; a graded free resolution is a free resolution of the underlying modules), so $\operatorname{Tor}_i^{\widehat{R}}(\widehat{M},\widehat{M})$ is nonzero for every i, whereas over the point all higher $\operatorname{Tor}_i^{\mathbb{K}_q}$ vanish.

It is natural to expect that the positivity and duality axioms pass to tensor products whenever the product is spectrally meaningful; this is the case.

Proposition 8.32. *Let D and $\widetilde{D}$ be Weil data of types $(d,0)$ and $(\tilde{d},0)$ satisfying the derivation rule of AW1, AW3, and AW4, and suppose that every $(V^a\otimes\widetilde{V}^b,\Theta\otimes 1+1\otimes\widetilde{\Theta})$ is a spectral core. Then the tensor datum $D\otimes\widetilde{D}$ (Definition 1.39) is a Weil datum of type $(d+\tilde{d},0)$ satisfying the derivation rule, AW3, and AW4, with star forms $h_{D\otimes\widetilde{D}}=\bigoplus_{a+b=n}h_a\otimes\tilde{h}_b$ in degree n; if D and $\widetilde{D}$ satisfy AW1 for $U=\mathbb{K}$, so does $D\otimes\widetilde{D}$.*

Proof. The Koszul product is associative, graded-commutative, and unital, and $\Theta\otimes 1+1\otimes\widetilde{\Theta}$ is a derivation of it. The trace is supported on bidegree $(2d,2\tilde{d})$, where the generator acts on $V^{2d}\otimes\widetilde{V}^{2\tilde{d}}$ with $\operatorname{Tr}(\Theta x)=d\operatorname{Tr}(x)$ and $\widetilde{\operatorname{Tr}}(\widetilde{\Theta}y)=\tilde{d}\widetilde{\operatorname{Tr}}(y)$, which gives trace equivariance into $\mathbb{C}(-d-\tilde{d})$. The duality pairing is $B(x\otimes y,x'\otimes y')=(-1)^{|y||x'|}B(x,x')\widetilde{B}(y,y')$ when $|x|+|x'|=2d$ and $|y|+|y'|=2\tilde{d}$, and zero otherwise; by AW3 for D and $\widetilde{D}$ and Proposition 8.3, it pairs the block $V^a_{\Theta\sim\rho}\otimes\widetilde{V}^b_{\Theta\sim\rho'}$ perfectly with $V^{2d-a}_{\Theta\sim d-\rho}\otimes\widetilde{V}^{2\tilde{d}-b}_{\Theta\sim\tilde{d}-\rho'}$ and vanishes on the other pairs of blocks, which by Proposition 8.2 is AW3 for $D\otimes\widetilde{D}$. For $|x|=a$ and $|y|=b$,

$$\Theta(*x\otimes\tilde{*}y)=(*\Theta x+(d-a)*x)\otimes\tilde{*}y+*x\otimes(\tilde{*}\widetilde{\Theta}y+(\tilde{d}-b)\tilde{*}y),$$

which with the constant sign $(-1)^{ab}$ is the star law in degree $a+b$. The star form of $x\otimes y$ and $x'\otimes y'$ vanishes unless both have bidegree (a,b), and then equals $(-1)^{ab}(-1)^{b(2d-a)}\operatorname{Tr}(x\cup *x')\widetilde{\operatorname{Tr}}(y\cup\tilde{*}y')=h_a(x,x')\tilde{h}_b(y,y')$. A tensor product of positive definite Hermitian forms is positive definite (in orthonormal bases the products of basis vectors are orthonormal). The endpoint conditions are $\mathbb{C}\otimes\mathbb{C}=\mathbb{C}$ and $\mathbb{C}(-d)\otimes\mathbb{C}(-\tilde{d})=\mathbb{C}(-d-\tilde{d})$. □

It is natural to expect the absolute product to be well behaved when one factor has finite-dimensional cohomology; for the Tate objects this is the case, and it is the Künneth reading of the identity $\zeta_{\mathbb{P}^1_{\mathbb{Z}}}(s)=\zeta(s)\zeta(s-1)$ (Proposition 7.27).

Proposition 8.33. *Let X be a scheme of finite type over $\mathbb{Z}$, $n\geq 0$, and ζ_X the Hasse–Weil zeta function of Section 7.6.*

(i) $\zeta_{X\times_{\mathbb{Z}}\mathbb{P}^n_{\mathbb{Z}}}(s)=\prod_{i=0}^n\zeta_X(s-i)$ on the half-plane where the Euler products converge absolutely.

(ii) If a Weil datum D of type $(d,0)$ satisfies the derivation rule, AW3, and AW4, then so does $D\otimes T_n$ (Definition 1.39); if D satisfies AW1 for $U=\mathbb{K}$, so does $D\otimes T_n$; if D satisfies AW2 for Z, then $D\otimes T_n$ satisfies AW2 for $Z_n(s)=\prod_{i=0}^n Z(s-i)$; and if D satisfies AW-det for Z with principal cuts and normalization term e^{a+bs}, then $D\otimes T_n$ satisfies AW-det for Z_n with principal cuts and normalization term $\prod_{i=0}^n e^{a+b(s-i)}$.

Proof of (i). Grouping closed points by residue characteristic, as in the proof of Proposition 7.27, $\zeta_X(s)=\prod_p Z(X_{\mathbb{F}_p},p^{-s})$. Since $\#\mathbb{P}^n(\mathbb{F}_{p^m})=\sum_{i=0}^n p^{mi}$, $\#(X\times\mathbb{P}^n)(\mathbb{F}_{p^m})=\sum_i p^{mi}\#X(\mathbb{F}_{p^m})$, and exponentiating gives $Z((X\times\mathbb{P}^n)_{\mathbb{F}_p},t)=\prod_i Z(X_{\mathbb{F}_p},p^i t)$; take the product over p with $t=p^{-s}$. □

Proof of (ii). T_n satisfies the derivation rule, AW1 for $U=\mathbb{K}$, AW3 (h^i pairs with h^{n-i}, eigenvalues $i+(n-i)=n$), and AW4 ($\Theta*h^i=(n-i)*h^i=*\Theta h^i+(n-2i)*h^i$ and $h_{2i}(h^i,h^i)=1$).

The spaces $V^a \otimes \mathbb{C}h^i = V^a(-i)$ are spectral cores, so Proposition 8.32 applies. In degree k the spectral divisor of $D \otimes T_n$ is $\sum_i \mathrm{Sp}_{k-2i}(D)$ shifted by i, and $(-1)^{k+1} = (-1)^{k-2i+1}$, so the alternating sum is $\sum_i$ of the divisor of $Z(s-i)$. For the principal cut, the spectral zeta function of $V(-i)$ at s is that of V at $s-i$, so its determinant is $D_j(s-i)$; spectral zeta functions of finite direct sums add on the intersection of the domains, which contains a right half-plane, so determinants multiply, and the alternating product is $\prod_i e^{a+b(s-i)} Z(s-i)$. □

8.5. **The Hodge index clause.** It is natural to expect that positivity on the arithmetic surface is a new condition, stronger or weaker than positivity on the curve; in fact, on the algebraic tensors, it is equivalent to purity together with definiteness of the star form on each eigenspace, and it is the positivity of an explicit involution on $\bigoplus_\rho \mathrm{End}(V_{K,\rho})$. It implies the algebra form AW4-E of the positivity axiom, and for a suitable star it follows from AW4-E. This is the exact counterpart of the positivity in Weil's proof: there the trace form of correspondences on $C \times C$ is positive (Castelnuovo; the Hodge index theorem), and this positivity is equivalent to that of the Rosati involution (Section 4.3). The deduction of $\mathrm{Tr}(u' \circ u) > 0$ from the Hodge index conjecture on $X \times X$ is Kleiman's [Kle68, 3.11, pp. 380–381], after Serre [Ser60, Théorème 2, p. 394]; the positivity of the Rosati involution is [Mum74, §21, Theorem 1, p. 192]. Theorem 8.34 is the arithmetic analogue of these statements.

Theorem 8.34. *Let K be a number field and let a Weil datum of type $(1,0)$ on $\overline{X}_K$ satisfy the derivation rule of AW1, AW3, and the algebraic part of AW4 in degree one (Definition 1.37), with the notation of Definition 8.11.*

(i) If $\mathrm{Re}\,\rho \neq \frac{1}{2}$, then h_1 vanishes on $V_{K,\rho} \times V_{K,\rho}$ and $h_\otimes$ vanishes on $V_{K,\rho} \otimes V_{K,1-\rho}$.

(ii) If $\mathrm{Re}\,\rho = \frac{1}{2}$ and $V_{K,\rho} \neq 0$, then h_1 is nondegenerate on $V_{K,\rho}$ and on $V_{K,1-\rho}$, of the same signature (p,q), and $h_\otimes$ is nondegenerate on $V_{K,\rho} \otimes V_{K,1-\rho}$ of signature $(p^2+q^2, 2pq)$.

(iii) $h_\otimes$ is positive definite on N_K if and only if every eigenvalue ρ of Θ_1 has $\mathrm{Re}\,\rho = \frac{1}{2}$ and h_1 is definite, positive or negative, on each $V_{K,\rho}$.

(iv) In that case Θ_1 is diagonalizable, the datum satisfies AW4-V and AW4-E in degree one, and if it also satisfies AW1 for $U = \mathbb{1}$ and AW2 for $(\Lambda_K, \mathbb{1})$, every zero of ξ_K has real part $\frac{1}{2}$.

(v) If S is a multiplicative surface datum over the datum and itself, then κ is injective on N_K and $h_S^2(\kappa u, \kappa u') = h_\otimes(u,u')$ for $u, u' \in V_K \otimes V_K$; so S satisfies the Hodge index clause if and only if $h_\otimes$ is positive definite on N_K.

(vi) If $\mathrm{Re}\,\rho = \frac{1}{2}$, then Φ_ρ is a linear isomorphism, $h_\otimes(u,u') = \mathrm{tr}(\Phi_\rho(u)\Phi_\rho(u')^\star)$ for $u, u' \in V_{K,\rho} \otimes V_{K,1-\rho}$, and $\Phi_\rho(\sum_i e_i \otimes f_i)$ is the identity whenever (e_i) is a basis of $V_{K,\rho}$ and (f_i) a basis of $V_{K,1-\rho}$ with $B_K(e_k, f_i) = \delta_{ki}$.

Proof of (i). An antilinear map commuting with Θ_1 sends $V_{K,\rho}$ into $V_{K,\bar\rho}$, since $(\Theta_1 - \bar\rho)^k *_1 = *_1 (\Theta_1 - \rho)^k$. So $h_1(x,y) = B_K(x, *_1 y)$ with $x \in V_{K,\rho}$ and $*_1 y \in V_{K,\bar\rho}$, which vanishes by Proposition 8.3 unless $\bar\rho = 1 - \rho$, that is $\mathrm{Re}\,\rho = \frac{1}{2}$. For $x, x' \in V_{K,\rho}$, $h_\otimes(x \otimes y, x' \otimes y') = h_1(x,x')h_1(y,y') = 0$. □

Proof of (ii). Now $\bar\rho = 1 - \rho$. The bijection $*_1$ maps $V_{K,\rho}$ into $V_{K,1-\rho}$ and $V_{K,1-\rho}$ into $V_{K,\rho}$, injectively, so it maps $V_{K,\rho}$ onto $V_{K,1-\rho}$. By AW3, B_K pairs $V_{K,\rho}$ and $V_{K,1-\rho}$ perfectly, so $h_1(x,y) = B_K(x, *_1 y)$ is nondegenerate on $V_{K,\rho}$, and likewise on $V_{K,1-\rho}$. For $a, b \in V_{K,\rho}$, $h_1(*_1 a, *_1 b) = B_K(*_1 a, *_1^2 b) = \overline{B_K(a, *_1 b)} = \overline{h_1(a,b)}$, so $*_1$ maps subspaces of $V_{K,\rho}$ on which h_1 is positive (negative) definite to subspaces of $V_{K,1-\rho}$ of the same dimension with the same property; the signatures agree. In bases orthogonal for the two restrictions, the products of basis vectors are $h_\otimes$-orthogonal with values the products of the signs, which gives $(pp+qq, pq+qp)$. □

Proof of (iii). If some eigenvalue has $\operatorname{Re}\rho \neq \frac{1}{2}$, then $V_{K,1-\rho} \neq 0$ by AW3 and the nonzero block $V_{K,\rho} \otimes V_{K,1-\rho}$ of N_K is $h_\otimes$-null by (i). If $\operatorname{Re}\rho = \frac{1}{2}$ and h_1 is indefinite on $V_{K,\rho}$, then $2pq > 0$ in (ii). Conversely, if all eigenvalues are on the line and h_1 is definite on each $V_{K,\rho}$, each block is positive definite by (ii), and distinct blocks are $h_\otimes$-orthogonal: $h_1(V_{K,\rho}, V_{K,\sigma}) = B_K(V_{K,\rho}, *_1 V_{K,\sigma})$ vanishes by Proposition 8.3 unless $\bar\sigma = 1-\rho$, that is $\sigma = \rho$. □

Proof of (iv). Let $\varepsilon_\rho = \pm 1$ make $\varepsilon_\rho h_1$ positive on $V_{K,\rho}$, and let h' be the orthogonal sum of the forms $\varepsilon_\rho h_1|_{V_{K,\rho}}$, a positive definite Hermitian form on V_K. The computation proving Proposition 6.1 for $j = 1$ uses only the derivation rule, the trace equivariance, and the star law in degree one, so h_1 satisfies (1.1) with weight one; since each $V_{K,\rho}$ is Θ_1-stable and distinct $V_{K,\rho}$ are h_1- and h'-orthogonal, so does h'. Thus AW4-V holds in degree one, AW4-E by Proposition 6.2(ii), diagonalizability by Proposition 3.3, and under AW1 and AW2 Theorem 6.3 gives the last assertion. □

Proof of (v). For $x, x', y, y' \in V_K$, multiplicativity of the star, associativity, and graded commutativity give $\kappa(x\otimes y)\cup *_S \kappa(x'\otimes y') = -\kappa(x\otimes y)\cup\kappa(*x'\otimes *y') = \mathrm{pr}_1^*(x\cup *x')\cup \mathrm{pr}_2^*(y\cup *y')$, and multiplicativity of the trace gives $h_S^2(\kappa(x\otimes y),\kappa(x'\otimes y')) = h_1(x,x')h_1(y,y')$. Injectivity is the proof of Theorem 8.12(i), which uses only the derivation rule, AW3, and the multiplicativity of the trace. □

Proof of (vi). Φ_ρ is injective because B_K pairs $V_{K,\rho}$ and $V_{K,1-\rho}$ perfectly, and both spaces have dimension $m(\rho)^2$. Write $y = *_1 y_0$ and $y' = *_1 y_0'$ with $y_0, y_0' \in V_{K,\rho}$. Then $\Phi_\rho(x\otimes y) = x\, h_1(\cdot, y_0)$, whose adjoint is $y_0\, h_1(\cdot, x)$, so $\operatorname{tr}(\Phi_\rho(x\otimes y)\Phi_\rho(x'\otimes y')^\star) = h_1(y_0', y_0)h_1(x,x')$; and $h_1(y,y') = \overline{h_1(y_0,y_0')} = h_1(y_0', y_0)$ by the computation in (ii). The last assertion is $\sum_i B_K(z, f_i)e_i = z$ for $z = \sum_k c_k e_k$. □

Thus on the algebraic tensors, identified by (vi) with $\bigoplus_\rho \operatorname{End}(V_{K,\rho})$, the Künneth form is the trace form $(u,u') \mapsto \operatorname{tr}(uu'^\star)$ of an involution, and the Hodge index clause is the positivity of that involution: the arithmetic form of Castelnuovo's positivity $\operatorname{Tr}(X\circ X') > 0$ on correspondences of $C\times C$ and of the Rosati positivity of Theorem 4.10(ii). The identity of $V_{K,\rho}$ for all ρ at once, the component of the diagonal, lies only in the product $\prod_\rho \operatorname{End}(V_{K,\rho})$, not in N_K: the diagonal is not an algebraic class of the algebraic tensor product, and its "self-intersection" $\sum_\rho m(\rho)$ diverges, the counterpart of $2g$ for a curve of infinite genus. The operators $\int f(t)\phi^t\,dt$ generated by the flow also lie only in that product. If all eigenvalues are on the line and h_1 is positive, such an operator acts on $V_{K,\rho}$ by the scalar $\int f(t)e^{t\rho}\,dt$, and the formal value of the trace form on it is $\sum_\rho m(\rho)|\int f(t)e^{t\rho}\,dt|^2$, the shape of the Weil form of Proposition 3.10(iii). Off the line the restriction carries no information: by (i) the form vanishes on the blocks of eigenvalues off the line, so the Hodge index clause on these operators alone is not the Weil positivity criterion.

It is natural to expect the converse of (iv), that AW4-E in degree one implies the Hodge index clause; in fact it fails for a given star, because AW4-E constrains the spectrum only, while the Hodge index clause also constrains the star.

Definition 8.35. Let $\rho = \frac{1}{2} + i\gamma$ with $\gamma > 0$. The *split-signature datum* at ρ is the Weil datum of type $(1,0)$ with $V^0 = \mathbb{C}\cdot 1$, $V^2 = \mathbb{C}\eta$, $\Theta_0 = 0$, $\Theta_2 = 1$, and V^1 with basis e_1, e_2, f_1, f_2, $\Theta_1 e_i = \rho e_i$, $\Theta_1 f_i = \bar\rho f_i$; with the product $u\cup v = B(u,v)\eta$ on V^1 for the alternating form with $B(e_i,f_k) = \delta_{ik} = -B(f_k,e_i)$ and $B(e_i,e_k) = B(f_i,f_k) = 0$, and the unit products otherwise; with the trace $\operatorname{Tr}(c\eta) = c$; and with the antilinear stars $*_0(c) = \bar c\eta$, $*_2(c\eta) = \bar c$, $*_1 e_i = \varepsilon_i f_i$, $*_1 f_i = -\varepsilon_i e_i$, where $(\varepsilon_1,\varepsilon_2) = (1,-1)$.

For this datum the notation of Definition 8.11 is used verbatim, with $V = V^1$, $V_\rho = \mathbb{C}e_1 \oplus \mathbb{C}e_2$, and $V_{\bar\rho} = \mathbb{C}f_1 \oplus \mathbb{C}f_2$; the definition there uses only a Weil datum of type $(1,0)$, not the field.

Proposition 8.36. *Let $\rho = \frac{1}{2} + i\gamma$ with $\gamma > 0$, and consider the split-signature datum at ρ (Definition 8.35). It satisfies AW1 for $U = \mathbb{1}\!\!\mathbb{F}$, AW3, the algebraic part of AW4 in degree one, AW4-V, and AW4-E in degree one, and $h_\otimes$ is not positive definite on $V_\rho \otimes V_{\bar\rho}$.*

Proof. AW1 and AW3 hold as for the trivial model (Proposition 7.6(i)), since B pairs the ρ- and $\bar\rho = (1-\rho)$-eigenspaces perfectly and vanishes otherwise. The map $*_1$ is bijective, sends V_ρ to $V_{\bar\rho}$ and back, so it commutes with Θ_1; on basis vectors $B(*_1 e_i, *_1 f_k) = -\varepsilon_i\varepsilon_k B(f_i, e_k) = \delta_{ik}$, $B(*_1 f_k, *_1 e_i) = -\delta_{ik}$, and the other values vanish, so $B(*_1 x, *_1 y) = \overline{B(x,y)}$ by antilinearity. The form $h_1(x,y) = B(x, *_1 y)$ has $h_1(e_i,e_k) = \varepsilon_k\delta_{ik}$, $h_1(f_i,f_k) = \varepsilon_k\delta_{ik}$, and $h_1(e_i,f_k) = h_1(f_k,e_i) = 0$, so it is Hermitian, of signature $(1,1)$ on V_ρ. The spectrum $\{\rho,\bar\rho\}$ lies on the line and Θ_1 is diagonal; the orthogonal sum of positive definite forms on the two eigenspaces satisfies (1.1) with weight one, since for eigenvectors with eigenvalues σ,σ' both sides are $(\sigma+\bar\sigma')h(u,v)$, which is $h(u,v)$ if $\sigma = \sigma'$ and 0 otherwise. So AW4-V holds, and AW4-E by Proposition 6.2(ii). By Theorem 8.34(ii), $h_\otimes$ has signature $(2,2)$ on $V_\rho \otimes V_{\bar\rho}$. □

With the star $*_1'$ of Proposition 6.2(iii) built from a positive polarization, the same datum satisfies AW4, and the tensor star form of $*_1'$ is positive definite, being the tensor square of a positive form: the Hodge index clause is a property of the star, not of the spectrum alone.

It is natural to expect the conjecture, like any existence statement about the surface, to have content beyond the Riemann hypothesis; in fact, without the requirement that the surface be constructed independently of the zeros, it collapses as the axioms on one object do.

Proposition 8.37. *Let K and L be number fields and D_K, D_L Weil data of type $(1,0)$ on $\overline{X}_K$ and $\overline{X}_L$ satisfying AW1 for $U = \mathbb{1}\!\!\mathbb{F}$.*

- *(i) The tensor datum $D_K \otimes D_L$ (Definition 1.39) with $\mathrm{pr}_1^* x = x \otimes 1$ and $\mathrm{pr}_2^* y = 1 \otimes y$ is a multiplicative surface datum over (D_K, D_L) whose Künneth map is the identity.*
- *(ii) The existence of the data of Conjecture 1.40 satisfying (K1) and (K2), with no condition on how they are constructed, is equivalent to the statement that every zero of every ξ_K has real part $\frac{1}{2}$.*

Proof of (i). By AW1 the products are associative, graded-commutative, and unital, and $\Theta(1) = 0$ because Θ is a derivation. The Koszul tensor product of such algebras is again associative, graded-commutative, and unital, $\Theta \otimes 1 + 1 \otimes \Theta$ is a derivation of it, the maps $x \mapsto x \otimes 1$ and $y \mapsto 1 \otimes y$ are unital algebra homomorphisms commuting with the flows (using $\Theta(1) = 0$), $(x \otimes 1) \cup (1 \otimes y) = x \otimes y$, the trace $\mathrm{Tr}_K \otimes \mathrm{Tr}_L$ is multiplicative, and the star rule is the definition of the stars in Definition 1.39. □

Proof of (ii). If every ξ_K has its zeros on the line, the trivial models with spectra $\operatorname{div}\xi_K$ satisfy AW1–AW4 (Proposition 7.7), and in particular the algebraic part of AW4 in degree one (the computation after Definition 1.37), with h_1 positive; take $S_{K,L} = D_K \otimes D_L$ as in (i). Its Künneth map is an isomorphism, and $h_\otimes = h_1 \otimes h_1$ is positive definite, so the Hodge index clause holds by Theorem 8.34(v). Conversely, the Hodge index clause for $S_{K,K}$ and Theorem 8.34(v), (iii), and (iv) put every zero of ξ_K on the line. □

Theorem 8.34 and Proposition 8.37 say that, as existence statements, the Hodge index clause and the positivity axiom on the curve carry the same content; for a given star the clause is the stronger condition (Proposition 8.36). What the arithmetic surface would add is a place where the positivity could be proved from geometry, as Weil proved Castelnuovo's inequality by intersection theory on $C \times C$; the statement to be proved is fixed by Theorem 8.34.

9. Negative results

This section records inferences that look plausible on the way to a Deninger cohomology theory or a de Branges realization of the zeros, but are not valid. Each finding is introduced by a sentence of the form "it is natural to expect X; in fact Y", followed by a numbered statement. Unless the contrary is said explicitly, a finding refutes an inference, not a theorem in the literature: the witnesses are counterexamples to deductions from the listed properties, not to the Weil conjectures, to the existence of any cohomology theory, or to any published main theorem. Section 9.5 maps the findings to a register of the expectations they correct.

9.1. **Algebraic findings.** The first two findings, R1 and R2 (Definitions 6.8 and 6.10, Propositions 6.9, 6.11, and 6.12), are stated in Section 6.1, before the main theorem, whose Section 7.3 uses them.

It is natural to expect that over a finite field a Frobenius commuting with a quaternionic structure and preserving a perfect alternating duality of weight one is pure; in fact the block of Definition 6.10 exponentiates to a counterexample.

Proposition 9.1. *Let $0 < a < \frac{1}{2}$, $T > 0$, $q > 1$, let Θ, Ω, and J be as in Definition 6.10, and let $\mathrm{F} = q^{\Theta} = \operatorname{diag}(q^{\lambda}, q^{\bar\lambda}, q^{1-\lambda}, q^{1-\bar\lambda})$. Then $\mathrm{F}^T\Omega\mathrm{F} = q\Omega$, $\mathrm{F}J = J\mathrm{F}$, $J^2 = -1$, and the eigenvalues of F have moduli $q^{1/2+a}$ and $q^{1/2-a}$.*

Proof. Θ is diagonal and $\Theta^T\Omega + \Omega\Theta = \Omega$ (proof of Proposition 6.11(i)), so the nonzero entries of Ω sit at pairs of eigenvalues with sum one, and $\mathrm{F}^T\Omega\mathrm{F}$ multiplies each of them by q. Since $J\Theta = \Theta J$ and J is antilinear, J maps the μ-eigenline of Θ to the $\bar\mu$-eigenline, on which F is $q^{\bar\mu} = \overline{q^{\mu}}$; so $J\mathrm{F} = \mathrm{F}J$. The equality $J^2 = -1$ is Proposition 6.11(i), and the moduli are $q^{\operatorname{Re}\mu}$. □

It is natural to expect that comparing V with its integer Tate twists by cones sees every summand, and that adjoining an acyclic block leaves determinant and special-value data unchanged; in fact integer-twist cones are blind to summands with nonreal spectrum, while determinants change by a nonconstant factor.

Definition 9.2. Let V be a finite-dimensional complex vector space, $\Theta : V \to V$ linear, and $n \in \mathbb{Z}$. The *twist cone* of (V, Θ) at n is the two-term complex $C_n(V, \Theta) = [V \xrightarrow{\Theta - n} V]$ in degrees 0 and 1.

Proposition 9.3. *Let $0 < a < \frac{1}{2}$, $T > 0$, and let (V, Θ), p, and w be as in Definition 6.10. Then:*

(i) for every $n \in \mathbb{Z}$ the map $\Theta - n$ is invertible, so $C_n(V, \Theta)$ has zero cohomology;

(ii) $p(s)$ equals

$$((w-a)^2 + T^2)((w+a)^2 + T^2) = w^4 + 2(T^2 - a^2)w^2 + (T^2 + a^2)^2,$$

satisfies $p(1-s) = p(s)$ and $p(s) > 0$ for real s, and has degree four;

(iii) if (V', Θ') is a spectral core that is zeta-regularizable for a spectral cut on an open set U, then $(V' \oplus V, \Theta' \oplus \Theta)$ is a spectral core that is zeta-regularizable for a spectral cut on a nonempty open subset $U_0 \subset U$ extending the given one, and, with the determinants taken for these two cuts,

$$\det_\infty\big((s - \Theta' \oplus \Theta)/2\pi\big) = (2\pi)^{-4} p(s) \det_\infty\big((s - \Theta')/2\pi\big).$$

Proof of (i). The eigenvalues $\frac{1}{2} \pm a \pm iT$ have nonzero imaginary part, so none equals n. □

Proof of (ii). $(s-\lambda)(s-\bar\lambda) = (w-a)^2 + T^2$ and $(s-(1-\lambda))(s-(1-\bar\lambda)) = (w+a)^2 + T^2$. With $A = w^2 + a^2 + T^2$ the product is $(A - 2aw)(A + 2aw) = A^2 - 4a^2w^2$, which expands to the

displayed quartic. The quartic is even in w, and $s \mapsto 1-s$ is $w \mapsto -w$. For real s both factors are at least $T^2 > 0$. □

Proof of (iii). The sum is a spectral core: its generalized eigenspaces are those of the summands added together, and its spectrum is the union of $\operatorname{spec}\Theta'$ with four points. Let Z' be the spectral zeta function of (V', Θ') for the given cut on U. Removing from U the four closed rays $\rho_k + (-\infty, 0]$ through the eigenvalues ρ_k of Θ leaves a nonempty open set U_0. Extend the cut to the sum by keeping the given branches on $\operatorname{spec}\Theta'$ and taking the principal branch of $\log((s-\rho_k)/2\pi)$ on U_0 for each $\rho_k \notin \operatorname{spec}\Theta'$; in either case the branch L_{ρ_k} satisfies $e^{L_{\rho_k}(s)} = (s-\rho_k)/2\pi$. On U_0 the spectral zeta function of the sum is $Z'(w;s) + \sum_{k=1}^{4} e^{-wL_{\rho_k}(s)}$; the added terms are entire in w, holomorphic in s, and have w-derivative $-L_{\rho_k}(s)$ at $w=0$. Hence

$$\exp\bigl(-\partial_w Z_{V'\oplus V}(0;s)\bigr) = \exp\bigl(-\partial_w Z'(0;s)\bigr) \prod_{k=1}^{4} \frac{s-\rho_k}{2\pi}$$

on U_0, and the right side continues to the entire function $(2\pi)^{-4}p(s)\det_\infty((s-\Theta')/2\pi)$. □

Thus the quasi-isomorphism classes of all integer-twist cones, and their long exact sequences, are unchanged by adjoining (V, Θ), whereas a degree-one determinant is multiplied by $(2\pi)^{-4}p(s)$, a nonconstant factor which is positive on the real axis and symmetric under $s \mapsto 1-s$. A determinant-line trivialization carrying metric or basis data can retain this factor even for an acyclic complex. The finding concerns integer-cone data alone; it is not a counterexample to the full package of Deninger and Flach–Morin [FM20, Conjecture 4.1], which also specifies determinant identities and leading coefficients.

It is natural to expect that for a higher-dimensional variety all zeros of its zeta function lie on a single critical line; in fact purity holds degree by degree, and zeros from different degrees lie on different lines.

Proposition 9.4. *Let q be a prime power, E an elliptic curve over $\mathbb{F}_q$ with Frobenius roots α, β, and $X = E \times \mathbb{P}^1$, with zeta functions $Z(E,t)$, $Z(X,t)$, and ζ_X as in Definition 3.7. Then:*

- *(i) $Z(X,t) = Z(E,t)\,Z(E,qt)$;*
- *(ii) the zeros of ζ_X form the set $\Sigma_q(\alpha) \cup \Sigma_q(\beta) \cup \bigl(1+\Sigma_q(\alpha)\bigr) \cup \bigl(1+\Sigma_q(\beta)\bigr)$ (Definition 4.14); each of the lines $\operatorname{Re} s = \frac{1}{2}$ and $\operatorname{Re} s = \frac{3}{2}$ contains infinitely many of them, and every zero lies on one of these two lines;*
- *(iii) the poles of ζ_X lie on the lines $\operatorname{Re} s \in \{0,1,2\}$.*

Proof of (i). $\#\mathbb{P}^1(\mathbb{F}_{q^n}) = q^n + 1$, so $\#X(\mathbb{F}_{q^n}) = \#E(\mathbb{F}_{q^n}) + q^n \#E(\mathbb{F}_{q^n})$. Exponentiating $\sum_n \#E(\mathbb{F}_{q^n})t^n/n$ gives $Z(E,t)$, and exponentiating $\sum_n \#E(\mathbb{F}_{q^n})(qt)^n/n$ gives $Z(E,qt)$. □

Proof of (ii) and (iii). By (3.1) (the case of elliptic curves of Theorem 3.8, Section 3.1), $Z(E,t) = (1-\alpha t)(1-\beta t)/((1-t)(1-qt))$ with $|\alpha| = |\beta| = q^{1/2}$. By (i),

$$\zeta_X(s) = \frac{(1-\alpha q^{-s})(1-\beta q^{-s})(1-\alpha q^{1-s})(1-\beta q^{1-s})}{(1-q^{-s})(1-q^{1-s})(1-q^{1-s})(1-q^{2-s})}.$$

By Proposition 4.16(i), $1-\alpha q^{-s}$ vanishes exactly on $\Sigma_q(\alpha)$, and $1-\alpha q^{1-s}$ vanishes exactly when $s-1 \in \Sigma_q(\alpha)$; likewise for β. Points of $\Sigma_q(\alpha)$ and $\Sigma_q(\beta)$ have real part $\frac{1}{2}$ since $|\alpha| = |\beta| = q^{1/2}$, and points of $1+\Sigma_q(\alpha)$ and $1+\Sigma_q(\beta)$ real part $\frac{3}{2}$; each of these sets is an infinite arithmetic progression. The denominator vanishes exactly where $q^s \in \{1, q, q^2\}$, that is on $\operatorname{Re} s \in \{0,1,2\}$. Since numerator and denominator have no common zero (their real parts differ), the zeros of ζ_X are those of the numerator and its poles those of the denominator. □

The zeros on $\operatorname{Re} s = \frac{1}{2}$ come from $H^1(X)$ (weight one) and those on $\operatorname{Re} s = \frac{3}{2}$ from $H^3(X)$ (weight three); this is the degree-by-degree form of purity, the finite-field counterpart of Theorem 7.28.

9.2. **Operator-theoretic and analytic findings.** It is natural to expect that a skew-symmetric operator has purely imaginary spectrum, so that the algebraic purity argument controls the whole spectrum of a completion; in fact a closed, densely defined skew-symmetric operator can have every complex number in its spectrum and no eigenvalues at all.

Definition 9.5. Let $H^1_0(0,1)$ be the space of absolutely continuous $f : [0,1] \to \mathbb{C}$ with $f' \in L^2(0,1)$ and $f(0) = f(1) = 0$. The *Dirichlet derivative* is the operator $Df = f'$ on $L^2(0,1)$ with domain $H^1_0(0,1)$.

Proposition 9.6. *Let D be the Dirichlet derivative of Definition 9.5. Then:*

(i) D is densely defined and closed;
(ii) $\langle Df, g\rangle = -\langle f, Dg\rangle$ for all $f, g \in H^1_0(0,1)$;
(iii) D has no eigenvalues;
(iv) for every $z \in \mathbb{C}$ the function $t \mapsto e^{zt}$ does not lie in the range of $D - z$; hence the spectrum of D is $\mathbb{C}$;
(v) D is not skew-adjoint.

Proof of (i). $C^\infty_c(0,1) \subset H^1_0(0,1)$ is dense in $L^2(0,1)$. Let $f_n \in H^1_0(0,1)$ with $f_n \to f$ and $f_n' \to g$ in L^2. Put $G(x) = \int_0^x g$. Since $f_n(0) = 0$, the Cauchy–Schwarz inequality gives $|f_n(x) - G(x)| \le \|f_n' - g\|_2$ for $x \in [0,1]$, so $f_n \to G$ uniformly. Hence $f = G$ almost everywhere, G is absolutely continuous with $G' = g \in L^2$, $G(0) = 0$, and $G(1) = \lim f_n(1) = 0$. Thus $f \in H^1_0(0,1)$ and $Df = g$. □

Proof of (ii). Integration by parts for absolutely continuous functions gives $\int_0^1 f'\bar g = [f\bar g]_0^1 - \int_0^1 f\bar g' = -\int_0^1 f\bar g'$. □

Proof of (iii). If $f' = zf$ with $f \in H^1_0(0,1)$, then $e^{-zt}f$ is absolutely continuous with derivative zero almost everywhere, hence constant, equal to its value $f(0) = 0$ at $t = 0$. □

Proof of (iv). If $(D - z)f = g$ with $f \in H^1_0(0,1)$, then $(e^{-zt}f)' = e^{-zt}g$, and integrating over $[0,1]$ gives $\int_0^1 e^{-zt}g(t)\,dt = e^{-z}f(1) - f(0) = 0$. For $g(t) = e^{zt}$ the integral is $\int_0^1 1\,dt = 1$. So $D - z$ is not surjective for any z, and every z lies in the spectrum. □

Proof of (v). If $D^* = -D$, then iD is self-adjoint, so $iD - i$ is surjective; then $D - 1 = -i(iD - i)$ is surjective, contradicting (iv) with $z = 1$. □

The eigenvector argument of Proposition 3.3 is algebraic and remains valid; what fails is any claim about the full spectrum of a completion obtained without a domain theorem. A self-adjoint realization requires such a statement, which is why AW5 asks for a strongly continuous unitary group with skew-adjoint generator.

It is natural to expect that a scalar de Branges model built from F retains the multiplicities of the zeros of F; in fact a real zero of order k becomes a common real zero of order $k - 1$ of E and $E^\#$, and a cyclic self-adjoint operator cannot carry multiplicity in its eigenspaces at all.

Definition 9.7. Let A be a self-adjoint operator on a Hilbert space $\mathscr{H}$. A vector $v_0 \in \mathscr{H}$ is *cyclic* for A if the closed linear span of $\{v_0\} \cup \{(A - z)^{-1}v_0 : z \in \mathbb{C} \setminus \mathbb{R}\}$ is $\mathscr{H}$.

Proposition 9.8.

(i) *Let G be an entire function with $G^{\#} = G$ and $G \not\equiv 0$, with HB companion $E = E_G$ (Definition 3.2). Then $E^{\#} = G - iG'$, and if $r \in \mathbb{R}$ is a zero of G of order $k \geq 1$, then E and $E^{\#}$ vanish at r to order exactly $k-1$ (in particular $E(r)E^{\#}(r) \neq 0$ when $k = 1$).*

(ii) *Let $G(z) = (z^2-1)^2$ and $E = E_G$. Then $E(z) = (z^2-1)(z^2-1+4iz)$ and $E^{\#}(z) = (z^2-1)(z^2-1-4iz)$, and ± 1 are common simple zeros of E and $E^{\#}$.*

(iii) *Let A be a self-adjoint operator on a Hilbert space $\mathscr{H}$ with a cyclic vector v_0. Then $\dim\ker(A-\lambda) \leq 1$ for every $\lambda \in \mathbb{C}$.*

Proof of (i). Since $(G')^{\#} = (G^{\#})' = G'$, we get $E^{\#} = G^{\#} - i(G')^{\#} = G - iG'$. Write $G = (z-r)^k U$ with U entire and $U(r) \neq 0$. Then $G' = (z-r)^{k-1}(kU + (z-r)U')$, so

$$E = (z-r)^{k-1}\big((z-r)U + ikU + i(z-r)U'\big),$$

and the bracket takes the value $ikU(r) \neq 0$ at r. The same computation with $-i$ in place of i applies to $E^{\#}$. □

Proof of (ii). $G'(z) = 4z(z^2-1)$, so $E = (z^2-1)(z^2-1+4iz)$ and, by (i), $E^{\#} = (z^2-1)(z^2-1-4iz)$. The second factors take the values $\pm 4i$ at $z = \pm 1$. □

Proof of (iii). Let $K = \ker(A-\lambda)$; we may assume $K \neq 0$, so $\lambda \in \mathbb{R}$. Let P be the orthogonal projection onto K and $R_z = (A-z)^{-1}$ for $z \notin \mathbb{R}$. For $v \in K$, $(A-z)v = (\lambda - z)v$, so $R_z v = (\lambda - z)^{-1} v \in K$. For $u \perp K$ and $v \in K$, $\langle R_z u, v\rangle = \langle u, R_{\bar z} v\rangle = 0$, using $R_z^* = R_{\bar z}$; so R_z preserves $K^{\perp}$ as well, and $PR_z = R_z P$. Hence $PR_z v_0 = R_z P v_0 = (\lambda - z)^{-1} P v_0 \in \mathbb{C}Pv_0$, and P maps the span of $\{v_0\} \cup \{R_z v_0\}$ into $\mathbb{C}Pv_0$. By continuity of P and cyclicity, $K = P\mathscr{H} \subset \mathbb{C}Pv_0$. □

For a positive measure $\mu = \sum_r m_r \delta_r$ on a discrete subset of $\mathbb{R}$, multiplication by the variable on $L^2(\mu)$ has, at each atom r, the eigenspace of functions supported on $\{r\}$, which is one-dimensional whatever the weight m_r: the residue of $-F'/F$ at r records m_r, the eigenspace dimension does not. In particular the zero operator on $\mathbb{C}^2$ admits no cyclic vector. By (i), the function $E = E_F = F + iF'$ of Proposition 12.1 may have common real zeros with $E^{\#}$ when F has multiple real zeros; requiring E to have no real zeros would assume simplicity. A scalar de Branges realization of arithmetic H^1 therefore needs structure beyond self-adjointness, and a multiplicity-preserving identification requires additional data.

It is natural to expect that the semilocal de Branges spaces of Connes–Consani–Moscovici may be identified with the global space attached to Ξ once both are known to be Hilbert spaces; in fact an abstract isomorphism of Hilbert spaces, unitary or merely bounded with bounded inverse, carries no spectral information.

Definition 9.9. For a finite set S of places of $\mathbb{Q}$ containing ∞ and $\lambda > 0$, we write $\mathcal{B}_\lambda^S$ for the Hilbert space of entire functions constructed in [CCM24, §4.8] from Sonin spaces, with its S-dependent induced norm, and υ_S for the isomorphism of hilbertian spaces (a linear homeomorphism) of [CCM24, Proposition 4.8].

Proposition 9.10. *Let $\mathscr{H}_1, \mathscr{H}_2$ be separable infinite-dimensional complex Hilbert spaces and let $(\lambda_n)_{n\geq 1}$, $(\mu_n)_{n\geq 1}$ be real sequences with $|\lambda_n| \to \infty$ and $|\mu_n| \to \infty$. Then:*

(i) *there is a unitary map $U : \mathscr{H}_1 \to \mathscr{H}_2$;*

(ii) *there are self-adjoint operators A_1 on $\mathscr{H}_1$ and A_2 on $\mathscr{H}_2$ with pure point spectrum such that $\dim\ker(A_1 - x) = \#\{n : \lambda_n = x\}$ and $\dim\ker(A_2 - x) = \#\{n : \mu_n = x\}$ for every $x \in \mathbb{R}$;*

(iii) *if $\#\{n : \lambda_n = x\} \neq \#\{n : \mu_n = x\}$ for some x, there is no bounded linear bijection $W : \mathscr{H}_1 \to \mathscr{H}_2$, unitary or not, with $W\mathcal{D}(A_1) = \mathcal{D}(A_2)$ and $WA_1 = A_2 W$ on $\mathcal{D}(A_1)$.*

Proof of (i). Choose orthonormal bases $(e_n)_{n\geq 1}$ of $\mathscr{H}_1$ and $(f_n)_{n\geq 1}$ of $\mathscr{H}_2$ and let $Ue_n = f_n$. □

Proof of (ii). Let $A_1 e_n = \lambda_n e_n$ with domain $\{\sum c_n e_n : \sum |\lambda_n c_n|^2 < \infty\}$, and similarly A_2. The argument in the proof of Proposition 6.5 shows that these diagonal operators are self-adjoint. A vector $\sum c_n e_n$ lies in $\ker(A_1 - x)$ if and only if $c_n(\lambda_n - x) = 0$ for all n, so the kernel is spanned by the e_n with $\lambda_n = x$, finitely many since $|\lambda_n| \to \infty$. The spectrum is the closure of $\{\lambda_n\}$, which is this discrete set. □

Proof of (iii). If $v \in \ker(A_1 - x)$, then $Wv \in \mathcal{D}(A_2)$ and $A_2 W v = W A_1 v = x W v$. If $u \in \ker(A_2 - x)$, then $W^{-1}u \in \mathcal{D}(A_1)$ and $W A_1 W^{-1} u = A_2 u = xu = W(xW^{-1}u)$, so $A_1 W^{-1} u = x W^{-1} u$ by injectivity of W. Hence W maps $\ker(A_1 - x)$ bijectively onto $\ker(A_2 - x)$, and the dimensions agree. □

The isomorphism υ_S of Definition 9.9 is a theorem of this abstract kind together with a compatibility with the amplification maps; the entire-function realization and norm of $\mathcal{B}_\lambda^S$ depend on S and λ. By Proposition 9.10, an identification with the global space of Section 12 would need an intertwiner of the specified operators, the same norm, a controlled limit in S and λ, and multiplicity data; none of these is supplied by [CCM24, Proposition 4.8].

It is natural to expect that positivity of $\operatorname{Im}(-F'/F)$ obtained high in the upper half-plane continues down to the real axis, or that an inverse canonical-system theorem can supply it; in fact positivity can hold on $\operatorname{Im} z > a$ and fail just below, and a single zero in $\mathbb{C}_+$ forces negative values next to it.

Definition 9.11. For $T > 0$ and $a > 0$, the *quartet* is the polynomial $P_{T,a}(z) = ((z - T)^2 + a^2)((z + T)^2 + a^2)$.

Proposition 9.12. *Let $T > 0$ and $a > 0$, and write $P = P_{T,a}$ (Definition 9.11).*

(i) P has real coefficients, $P(-z) = P(z)$, and its zeros $\pm T \pm ia$ are not real.

(ii) $\operatorname{Im}(-P'(z)/P(z)) > 0$ for every z with $\operatorname{Im} z > a$.

(iii) If $T = 2$ and $a = \frac{1}{4}$, then

$$\operatorname{Im}\Big(-\frac{P'(2+\frac{i}{8})}{P(2+\frac{i}{8})}\Big) = -\frac{16892192}{3176475},$$

and consequently the HB companion E_P (Definition 3.2) satisfies $|E_P(2+\frac{i}{8})| < |E_P^{\#}(2+\frac{i}{8})|$.

(iv) Let G be an entire function, $G \not\equiv 0$, with a zero $z_0 \in \mathbb{C}_+$ of order $k \geq 1$. Then there is $\varepsilon_0 \in (0, \operatorname{Im} z_0)$ such that $-G'/G$ is holomorphic at $z_0 - i\varepsilon$ and $\operatorname{Im}(-G'/G)(z_0 - i\varepsilon) < 0$ for $0 < \varepsilon < \varepsilon_0$.

Proof of (i). Both factors are real polynomials, and $z \mapsto -z$ interchanges them. □

Proof of (ii). With r running over the four zeros, $-P'/P = \sum_r 1/(r - z)$ and $\operatorname{Im}\big(1/(r-z)\big) = (\operatorname{Im} z - \operatorname{Im} r)/|z - r|^2$. If $\operatorname{Im} z > a$, every term is positive, since $\operatorname{Im} r = \pm a$. □

Proof of (iii). At $z = 2 + \frac{i}{8}$ the four terms of the preceding proof are, for $r = 2 + \frac{i}{4}$, $2 - \frac{i}{4}$, $-2 + \frac{i}{4}$, $-2 - \frac{i}{4}$:

$$\frac{-1/8}{1/64} = -8, \quad \frac{3/8}{9/64} = \frac{8}{3}, \quad \frac{-1/8}{16 + 1/64} = -\frac{8}{1025}, \quad \frac{3/8}{16 + 9/64} = \frac{24}{1033}.$$

Their sum is $-16892192/3176475$. Since $P^{\#} = P$, $E_P^{\#} = P - iP'$, and $|x+y|^2 - |x-y|^2 = 4\operatorname{Re}(x\bar{y})$ with $x = P$, $y = iP'$ gives $|E_P|^2 - |E_P^{\#}|^2 = 4\operatorname{Im}(P\overline{P'}) = 4|P|^2 \operatorname{Im}(-P'/P)$, which is negative at $2 + \frac{i}{8}$. □

Proof of (iv). Near z_0, $-G'/G = -k/(z-z_0) + g(z)$ with g holomorphic on a disc $|z-z_0| < \delta$ containing no other zero of G, and $|g| \le M$ on $|z-z_0| \le \delta/2$. At $z = z_0 - i\varepsilon$, $-k/(z-z_0) = -ik/\varepsilon$, so $\operatorname{Im}(-G'/G) \le -k/\varepsilon + M < 0$ once $\varepsilon < k/M$. Take $\varepsilon_0 = \min(\delta/2, k/M, \operatorname{Im} z_0/2)$ (with $k/M = \infty$ if $M = 0$); then $z_0 - i\varepsilon \in \mathbb{C}_+$ for $0 < \varepsilon < \varepsilon_0$. □

By Proposition 12.1, RH is equivalent to positivity of $\operatorname{Im}(-F'/F)$ on $\mathbb{C}_+$, and to the HB inequality for $E_F = F + iF'$. The quartet P has the symmetries of Ξ and is positive in this sense above its zeros, but not below. An inverse canonical-system theorem whose input already contains the HB inequality for $\Xi + i\Xi'$ cannot establish that inequality; positivity, and the relevant endpoint condition, must be obtained independently.

It is natural to expect that an entire function which is the Fourier transform of a positive measure has only real zeros; in fact positivity of the kernel alone does not locate zeros.

Definition 9.13. (a) The *Fourier transform* of a finite positive Borel measure ν on $\mathbb{R}$ with compact support is the entire function $\widehat{\nu}(z) = \int e^{izt}\, d\nu(t)$ (the convention of Definition 3.9).

(b) For $a > 0$, put $\mu_a = \cosh(a)\delta_0 + \frac{1}{2}(\delta_1 + \delta_{-1})$.

(c) For $a > 0$ and $\sigma > 0$, put $g_\sigma(t) = (4\pi\sigma)^{-1/2} e^{-t^2/(4\sigma)}$ and $k_{a,\sigma}(t) = \cosh(a) g_\sigma(t) + \frac{1}{2}(g_\sigma(t-1) + g_\sigma(t+1))$.

Proposition 9.14. *Let $a > 0$, and let μ_a, g_σ, and $k_{a,\sigma}$ be as in Definition 9.13.*

(i) $\widehat{\mu_a}(z) = \cos z + \cosh a$ for all $z \in \mathbb{C}$, and its zeros are the points $(2m+1)\pi \pm ia$, $m \in \mathbb{Z}$, all simple and none real.

(ii) For every $\sigma > 0$, $k_{a,\sigma}$ is smooth, integrable, and everywhere positive, and $\int e^{izt} k_{a,\sigma}(t)\, dt = e^{-\sigma z^2}(\cos z + \cosh a)$ for all $z \in \mathbb{C}$; in particular it has the same zeros as $\widehat{\mu_a}$.

Proof of (i). $\widehat{\mu_a}(z) = \cosh a + \frac{1}{2}(e^{iz} + e^{-iz})$. Since $\cos(\pi + ia) = -\cos(ia) = -\cosh a$, and $\cos z = \cos z_0$ if and only if $z \in \pm z_0 + 2\pi\mathbb{Z}$, the zeros are $\pm(\pi + ia) + 2\pi\mathbb{Z}$, which is the displayed set. The derivative $-\sin z$ takes the values $\pm\sin(ia) = \pm i \sinh a \neq 0$ at $\pm(\pi + ia)$, and by periodicity at all zeros. □

Proof of (ii). Positivity, smoothness, and integrability are clear. For real x, $\int e^{ixt} g_\sigma(t)\, dt = e^{-\sigma x^2}$ (the Gaussian integral); both sides are entire in x, since g_σ has Gaussian decay, so the identity holds for all $z \in \mathbb{C}$. A translate $g_\sigma(t \mp 1)$ has transform $e^{\pm iz} e^{-\sigma z^2}$, and summing gives $e^{-\sigma z^2}\widehat{\mu_a}(z)$. The factor $e^{-\sigma z^2}$ has no zeros. □

For $0 < a < \frac{1}{2}$ the zeros lie in the strip $|\operatorname{Im} z| < \frac{1}{2}$, which corresponds to the critical strip under $z = i(s - \frac{1}{2})$. These examples refute the inference from positivity of the kernel alone; they do not reproduce the further identities satisfied by the theta kernel of ζ.

It is natural to expect that Weil positivity proved for tests with small support extends to all supports, that a compact perturbation of a positive operator stays positive, and that positivity of a shifted form $Q - \varepsilon_{\min} I$ on a quotient gives positivity of Q; in fact none of these passages is valid without further input.

Definition 9.15. Let $f \in C_c(\mathbb{R})$. Its *correlation* is $c_f(x) = \int_{\mathbb{R}} f(y)\overline{f(y-x)}\, dy$. The *window hypotheses* of [CC20, Theorem 1] are: multiplicative support of the test function in $[2^{-1/2}, 2^{1/2}]$, that is additive support in $[-\frac{1}{2}\log 2, \frac{1}{2}\log 2]$, and vanishing of its transform at $i/2$ and at 0. The *lowest eigenvector hypotheses* of [CCM25, Theorem 5.10] (version 1) are: the lowest eigenvalue $\varepsilon_{\min}$ of the finite Weil matrix is simple, it has an even eigenvector v, and a boundary functional δ_N satisfies $\delta_N(v) \neq 0$ (normalized there to $\delta_N(v) = 1$); the construction uses the inner product defined by $Q - \varepsilon_{\min} I$ on a quotient.

Proposition 9.16.

(i) *Let $\mathscr{H}$ be a Hilbert space and P an orthogonal projection of rank one. Then $-2P$ is compact and $I-2P$ has the eigenvalue -1.*

(ii) *For every $\varepsilon<0$ there is a bounded self-adjoint operator Q on a Hilbert space such that $Q-\varepsilon I$ is positive semidefinite and Q is not.*

(iii) *Let $\ell>0$ and $f\in C_c(\mathbb{R})$ with $\operatorname{supp} f\subset[-\ell,\ell]$. Then $c_f(x)=0$ for $|x|\geq 2\ell$. If moreover f is real and $f>0$ on $(-\ell,\ell)$, then $c_f(x)>0$ for $|x|<2\ell$.*

(iv) *Let $\ell>0$. If $\ell\leq\frac{1}{2}\log 2$, then $c_f(\log n)=0$ for every f as in (iii) and every integer $n\geq 2$. If $\ell>\frac{1}{2}\log 2$, then there is f as in (iii) with $c_f(\log n)>0$ for every integer n with $2\leq n<e^{2\ell}$.*

Proof of (i). $-2P$ has rank one. If $Pv=v\neq 0$, then $(I-2P)v=-v$. □

Proof of (ii). Take $\mathscr{H}=\mathbb{C}$ and $Q=\varepsilon$; then $Q-\varepsilon I=0$ and $Q<0$. □

Proof of (iii). The integrand $f(y)\overline{f(y-x)}$ vanishes unless $y\in[-\ell,\ell]$ and $y\in[x-\ell,x+\ell]$. For $|x|\geq 2\ell$ this set has measure zero. For $|x|<2\ell$ and f real and positive on $(-\ell,\ell)$, the integrand is continuous, nonnegative, and positive on the nonempty open interval $(-\ell,\ell)\cap(x-\ell,x+\ell)$. □

Proof of (iv). If $\ell\leq\frac{1}{2}\log 2$ and $n\geq 2$, then $\log n\geq 2\ell$; apply (iii). If $\ell>\frac{1}{2}\log 2$, take $f(t)=\max(0,\ell-|t|)$, which is real and positive on $(-\ell,\ell)$; for $2\leq n<e^{2\ell}$ we have $0<\log n<2\ell$. □

The values $c_f(\log n)$, $n\geq 2$ a prime power, are those at which the prime-power terms of Weil's explicit formula (Theorem 11.28) evaluate the correlation of a test function. By (iv), the window hypotheses of Definition 9.15 are exactly the range in which no such term is seen; enlarging the support brings in every prime power below $e^{2\ell}$. Positivity of the Weil form on test functions supported in this prime-free window, which then involves only the archimedean term, is due to Yoshida [Yos92, Theorem 1, p. 310], and was reproved by Bombieri [Bom00, §12, Theorem 12, pp. 226–227] and Burnol [Bur00, Theorem, p. 2]; compare [CC20, p. 2 and (149), p. 50]. The positivity conclusion of [CC20, Theorem 1] is proved under those hypotheses, and its proof uses the absence of prime-power terms, which fails for larger supports; extending the conclusion to larger supports requires new input. Items (i) and (ii) apply to the construction of [CCM25, Theorem 5.10] (version 1): the form $Q-\varepsilon_{\min}I$ is positive on its quotient even when $\varepsilon_{\min}<0$, and nothing about positivity of the Weil form itself follows. That construction is carried here with its lowest eigenvector hypotheses; no assertion is made about later versions.

The short support in Definition 9.15 cannot simply be removed even for the archimedean contribution alone. In the centered convention of [CC20, (152)–(153), Appendix B], put

$$\widehat{g}(t)=\int_0^\infty g(u)u^{-it}\,\frac{du}{u},\qquad m(t)=\operatorname{Re}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{it}{2}\right)-\log\pi.$$

Then $W_\infty(g\star g^*)=(2\pi)^{-1}\int_{\mathbb{R}}m(t)|\widehat{g}(t)|^2\,dt$, where $g^*(u)=\overline{g(u^{-1})}$ and $W_\infty=-W_{\mathbb{R}}$ in that source.

Proposition 9.17. *There are smooth compactly supported g on $\mathbb{R}_+^\times$ with $\widehat{g}(0)=\widehat{g}(i/2)=\widehat{g}(-i/2)=0$ and $W_\infty(g\star g^*)<0$.*

Proof. Choose nonzero $\phi\in C_c^\infty((-1,1))$ with $\int\phi=0$ and set

$$a_L(x)=L^{-1/2}\phi(x/L),\qquad g_L(e^x)=(1-4\partial_x^2)a_L(x).$$

Thus $\operatorname{supp} g_L \subset [e^{-L}, e^L]$ and $\widehat{g_L}(t) = (1+4t^2)\sqrt{L}\,\widehat{\phi}(Lt)$, also for complex t. The three asserted zeros follow. Substituting $y = Lt$,

$$W_\infty(g_L \star g_L^*) = \frac{1}{2\pi}\int_{\mathbb{R}} m(y/L)(1+4y^2/L^2)^2|\widehat{\phi}(y)|^2\,dy \longrightarrow m(0)\|\phi\|_2^2.$$

The convergence follows from the logarithmic growth of m and rapid decay of $\widehat{\phi}$. The digamma value at $1/4$ gives $m(0) = -\gamma - \pi/2 - 3\log 2 - \log\pi < 0$. □

This is a failure of positivity for the single archimedean term on a larger test class. The full Weil form also contains prime terms; this proposition neither contradicts the short-window theorem nor gives an off-line zero of ζ.

It is natural to expect that convergence of trial functions, a small residual, or a small L^2 error on growing supports identifies the eigentransforms required by a spectral approach; in fact each of these is compatible with failure of the intended conclusion.

Definition 9.18. For $g \in L^2(\mathbb{R})$ with compact support, its *negative-sign Fourier transform* is the entire function $\mathcal{F}_- g(z) = \int_{\mathbb{R}} g(t)e^{-izt}\,dt$.

Definition 9.19. (a) For $\delta > 0$, let $A_\delta = \operatorname{diag}(0,\delta,1)$ on $\mathbb{C}^3$, $k = (1,1,0)$, and P_δ the orthogonal projection onto $\ker A_\delta$.

(b) For $b > 0$ and $\ell \geq 1$, let $f_\ell = e^{-b\ell}(1_{[\ell-1,\ell]} + 1_{[-\ell,-\ell+1]})$.

(c) Let A be the operator $Ae_n = n^2 e_n$ on $\ell^2(\mathbb{N})$, $n \geq 1$, with domain $\{\sum c_n e_n : \sum n^4|c_n|^2 < \infty\}$.

Proposition 9.20. *Let A_δ, k, P_δ, f_ℓ, and A be as in Definition 9.19.*

(i) For every $\delta > 0$, $\|A_\delta k\| = \delta$ and $\|k - P_\delta k\| = 1$.

(ii) Let $b > 0$ and $\ell \geq 1$. Then f_ℓ is even, $\|f_\ell\|_2 = \sqrt{2}e^{-b\ell}$, and

$$\mathcal{F}_- f_\ell(ib) = \frac{1-e^{-b}}{b} + e^{-2b\ell}\frac{e^b-1}{b} \longrightarrow \frac{1-e^{-b}}{b} > 0 \quad (\ell \to \infty).$$

Moreover, for every z with $\operatorname{Im} z = b$, $\sup\{|\mathcal{F}_- g(z)| : \operatorname{supp} g \subset [-\ell,\ell],\ \|g\|_2 \leq 1\} = \sqrt{\sinh(2b\ell)/b}$.

(iii) The vectors $x_n = e_n/n$ satisfy $\|x_n\| \to 0$ and $\|Ax_n\| = n$.

Proof of (i). $A_\delta k = (0,\delta,0)$. Since $\delta > 0$, $\ker A_\delta = \mathbb{C}(1,0,0)$, so $P_\delta k = (1,0,0)$ and $k - P_\delta k = (0,1,0)$. □

Proof of (ii). Evenness and the norm are immediate. Since $e^{-i(ib)t} = e^{bt}$,

$$\mathcal{F}_- f_\ell(ib) = e^{-b\ell}\Big(\int_{\ell-1}^{\ell} e^{bt}dt + \int_{-\ell}^{-\ell+1} e^{bt}dt\Big) = \frac{1-e^{-b}}{b} + e^{-2b\ell}\frac{e^b-1}{b}.$$

For $\operatorname{Im} z = b$, $|e^{-izt}| = e^{bt}$, so by the Cauchy–Schwarz inequality $|\mathcal{F}_- g(z)| \leq \|g\|_2(\int_{-\ell}^{\ell} e^{2bt}dt)^{1/2}$, with equality for g proportional to $\overline{e^{-izt}}$ on $[-\ell,\ell]$; and $\int_{-\ell}^{\ell} e^{2bt}dt = \sinh(2b\ell)/b$. □

Proof of (iii). $\|x_n\| = 1/n$ and $Ax_n = ne_n$. □

In (i) the residual tends to zero while the distance to the target line does not; the ratio of residual to spectral gap, here exactly one, is the relevant quantity. In (ii), unweighted L^2 convergence on growing supports does not control transforms off the real axis; the evaluation norm grows like $e^{b\ell}$. In (iii), approximation of vectors does not control an unbounded operator without graph-norm estimates. A locally uniform nonzero limit of entire functions with only real zeros has only real zeros (Hurwitz); finite families of trial functions with real zeros do not by themselves supply such a limit.

9.3. **Coefficients and the local–global passage.** It is natural to expect that Deninger's obstruction to real coefficients is either irrelevant to the axioms or rules out every candidate; in fact it is a theorem with precise hypotheses, and a separate algebraic obstruction already appears over finite fields.

Remark 9.21. By Theorem 11.1, imported from [Den23, Theorem 2.13], no functorial theory with real coefficients on Arakelov compactifications of spectra of rings of integers satisfies all of the properties 2.1–2.12 of [Den23]. The proof there uses a quaternionic Galois extension with a symplectic irreducible representation of root number -1, whose central L-order is odd, whereas a symplectic irreducible occurs with even multiplicity in a complexified real representation. It excludes the naive untwisted real leafwise candidate only insofar as that candidate satisfies all of the properties 2.1–2.12. It does not exclude complex theories, quaternionic coefficient targets, the real Weil–Arakelov complexes of [FM20, §3], or a self-adjoint operator attached to ζ alone; Deninger himself notes that the analogy requires a twisted complex version [Den23, p. 6].

A separate algebraic obstruction appears over finite fields.

Proposition 9.22 (Serre)**.** *Let $\mathbb{H}$ be Hamilton's quaternion algebra over $\mathbb{R}$. There is no unital ring antihomomorphism $\mathbb{H} \to M_2(\mathbb{R})$.*

Proof. Let φ be such a map and put $\psi(x) = \varphi(x)^T$; then ψ is a unital ring homomorphism, since $\psi(xy) = (\varphi(y)\varphi(x))^T = \psi(x)\psi(y)$. Put $X = \psi(i)$ and $Y = \psi(j)$. Additivity and $\psi(1) = I$ give $\psi(-1) = -I$, so $X^2 = Y^2 = -I$ and $XY = \psi(ij) = \psi(-ji) = -YX$. The $\mathbb{R}$-algebra $\mathbb{H}$ has basis $1, i, j, ij$ with multiplication table determined by $i^2 = j^2 = -1$ and $ij = -ji$; the matrices I, X, Y, XY satisfy the same relations, hence the same table. So the $\mathbb{R}$-linear map $\chi : \mathbb{H} \to M_2(\mathbb{R})$, $\chi(c_0 + c_1 i + c_2 j + c_3 ij) = c_0 I + c_1 X + c_2 Y + c_3 XY$, is a unital $\mathbb{R}$-algebra homomorphism. Its kernel is a two-sided ideal not containing 1; as $\mathbb{H}$ is a division ring, χ is injective, hence bijective since both sides have dimension four. Then $M_2(\mathbb{R})$ would be a division ring, but $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)^2 = 0$. □

Proposition 9.22 is the algebraic core of Serre's observation that a Weil cohomology cannot have coefficients in $\mathbb{R}$ (or $\mathbb{Q}$, or $\mathbb{Q}_p$) [Mil15, (2.2) and fn. 25, p. 27], [Den23, §1, p. 1]: for a supersingular elliptic curve over $\overline{\mathbb{F}}_p$ the endomorphism algebra tensored with $\mathbb{R}$ is $\mathbb{H}$ (Deuring; see [Sil09, Theorem V.3.1, pp. 144–145]), and a contravariant functorial two-dimensional real H^1 would give a unital antihomomorphism $\mathbb{H} \to M_2(\mathbb{R})$. This is why the choice of coefficient category is substantive.

It is natural to expect that reduced leafwise cohomology of a foliation is represented by leafwise harmonic forms, as in Hodge theory, and that dynamical traces pass to it; in fact this fails without geometric hypotheses.

Definition 9.23. Let $\mathcal{F}$ be a smooth foliation of a closed manifold M with leafwise de Rham complex $(\mathcal{A}^\bullet_{\mathcal{F}}(M), d_{\mathcal{F}})$ of smooth leafwise forms. The *reduced leafwise cohomology* is $\overline{H}^j_{\mathcal{F}}(M) = \ker d_{\mathcal{F}} / \overline{\operatorname{im} d_{\mathcal{F}}}$, the closure being taken in the smooth Fréchet topology. Given a leafwise metric, a leafwise form ω is *harmonic* if $d_{\mathcal{F}}\omega = 0 = d^*_{\mathcal{F}}\omega$.

Remark 9.24. By Theorem 11.2, imported from [DS01, Proposition 1.2 and (12), p. 214], there exist a closed three-dimensional Heisenberg nilmanifold M, a smooth one-dimensional foliation $\mathcal{F}$ of M, and a leafwise metric such that $\overline{H}^1_{\mathcal{F}}(M)$ (Definition 9.23) is infinite-dimensional while the space of smooth leafwise harmonic 1-forms is one-dimensional.

In the source, §2 also obstructs a general passage of dynamical traces to reduced cohomology. The positive Hodge-type results for foliations assume a Riemannian foliation with a bundle-like

metric. A leafwise candidate for AW4–AW5 must therefore specify its topology and geometric hypotheses.

It is natural to expect that local determinant formulas at the finite and infinite places glue to a global cohomology realizing ξ; in fact the simplest gluing, the direct sum of local spectral spaces, keeps the local divisors, which are disjoint from the zeros of ξ.

Definition 9.25. For a prime p, the *local spectral core at p* is $V_p = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}e_{p,n}$ with $\Theta_p e_{p,n} = (2\pi i n/\log p)e_{p,n}$; it is the logarithmic suspension (Definition 4.15) of $V = \mathbb{C}$, $\mathrm{F} = 1$, with $q = p$. For a set $\mathcal{Q}$ of primes let $V_{\mathcal{Q}} = \bigoplus_{p\in\mathcal{Q}} V_p$ (algebraic direct sum) with $\Theta_{\mathcal{Q}} = \bigoplus_p \Theta_p$.

Proposition 9.26. *Let $\mathcal{Q}$ be a nonempty set of primes and let $(V_{\mathcal{Q}}, \Theta_{\mathcal{Q}})$ be as in Definition 9.25.*

(i) For each p, the zeros of $1 - p^{-s}$ are exactly the eigenvalues of Θ_p, each simple. The eigenvalues of $\Theta_{\mathcal{Q}}$ form $\bigcup_{p\in\mathcal{Q}}(2\pi i/\log p)\mathbb{Z} \subset i\mathbb{R}$; the eigenvalue 0 has multiplicity $\#\mathcal{Q}$ and every nonzero eigenvalue is simple.

(ii) If $\mathcal{Q}$ is infinite, $(V_{\mathcal{Q}}, \Theta_{\mathcal{Q}})$ is not a spectral core.

(iii) No eigenvalue of $\Theta_{\mathcal{Q}}$ is a zero of ξ.

(iv) Let $s \in \mathbb{C}$ with $p^s \neq 1$ for every prime p. The series $\sum_p |(1 - p^{-s})^{-1} - 1|$ converges if and only if $\operatorname{Re} s > 1$.

Proof of (i). The first assertion is Proposition 4.16(i) with $\alpha = 1$. If $2\pi i n/\log p = 2\pi i m/\log q$ with primes $p \neq q$, then $n \log q = m \log p$, so $q^n = p^m$, and unique factorization forces $n = m = 0$. □

Proof of (ii). The generalized eigenspace at 0 contains the linearly independent vectors $e_{p,0}$, $p \in \mathcal{Q}$, so it is infinite-dimensional. □

Proof of (iii). The eigenvalues lie on $\operatorname{Re} s = 0$, while the zeros of ξ are the nontrivial zeros of ζ, which lie in $0 < \operatorname{Re} s < 1$ (Theorem 3.4). □

Proof of (iv). Write $\sigma = \operatorname{Re} s$ and $a_p = (1 - p^{-s})^{-1} - 1 = 1/(p^s - 1)$, so $|p^s| = p^\sigma$. If $\sigma > 1$, then $|a_p| \le 1/(p^\sigma - 1) \le 2p^{-\sigma}$ and $\sum_p p^{-\sigma}$ converges. If $0 < \sigma \le 1$, then $|a_p| \ge 1/(p^\sigma + 1) \ge p^{-\sigma}/2 \ge 1/(2p)$, and $\sum_p 1/p$ diverges (Euler). If $\sigma \le 0$, then $|a_p| \ge 1/(p^\sigma + 1) \ge \frac{1}{2}$. □

Proposition 9.26 concerns only the direct-sum construction. The general statement that local determinant formulas do not glue by themselves is a scope remark: Hesselholt's formula [Hes18, Theorem A] concerns smooth proper schemes over one finite field, and the archimedean cyclic formula [CC14b, Theorem 1.1] concerns gamma factors of a smooth projective variety at one infinite place. Their categories, gradings, coefficient rings, and operators differ, and neither supplies maps identifying local objects, a descent datum, or a global positive pairing. By (iv), the global divisor of ζ arises only after analytic continuation of a product which converges absolutely only for $\operatorname{Re} s > 1$; interchanging infinite determinant products with analytic continuation requires a theorem. A restricted product, a descent complex, or a quotient might be meaningful, but none is specified by the analogy.

Remark 9.27. No composition of the comparison theorems 11.3, 11.5, 11.6, 11.7, 11.8, 11.9, 11.12, and 11.14 produces the comparison asked for in Question 12.15. Their sources are, respectively: smooth proper schemes over a finite field; proper smooth formal schemes over $\mathcal{O}_C$; p-adic completions of smooth $\mathcal{O}_C$-algebras, and quasisyntomic rings; smooth p-adic formal schemes over A/I for a bounded prism (A, I); smooth projective varieties over a number field, at one infinite place; smooth schemes over the residue field of a mixed-characteristic valued field; differential modules over a differential field of characteristic zero; and arc-motives over an algebraically closed nondiscrete Banach field. A composite of comparison maps has the source

of its first map, so none takes $\overline{X}_{\mathbb{Q}}$ with a flow as input, and none outputs a Hilbert space with an operator whose spectrum is the divisor of ξ. New constructions are needed on both ends.

It is natural to expect that Ayoub's conservativity theorem is available as a bridge among realizations; in fact the author's own paper list records a gap in it.

Remark 9.28. The author's list of papers [Ayo] user.math.uzh.ch/ayoub identifies a gap in [Ayo18, Theorem 8.3.10] and treats the conservativity conjecture as open pending a correction. Nothing in these notes uses that statement. This does not affect the results imported as Theorems 11.9 and 11.12.

9.4. **Findings about specific versions of the literature.** It is natural to expect that de Branges's positivity conditions for the space attached to ξ hold, as would follow from a successful execution of his programme; in fact Conrey and Li exhibit explicit failures.

Definition 9.29. For an entire function E satisfying the HB inequality, the *de Branges space* $\mathcal{H}(E)$ is the space of entire G with G/E, $G^{\#}/E \in H^2(\mathbb{C}_+)$, normed by $\|G\|^2 = \int_{\mathbb{R}} |G/E|^2\,dx$. The *de Branges positivity conditions* are the shift-positivity conditions for $\mathcal{H}(E)$ with $E(z) = \xi(1 - iz)$, and for the associated space of reciprocal functions, as formulated in [CL00, §3.1, (3.1)–(3.4), pp. 6–7].

Remark 9.30. Conrey and Li show, by high-precision numerical evaluation, that the de Branges positivity conditions of Definition 9.29 do not all hold (Theorem 11.29). Their witnesses in [CL00, §3.1] are numerical values, not interval-certified bounds, and the conclusion is recorded here at that strength. For the condition on the reciprocal space they also report a proof by Sarnak that involves no numerical calculation [CL00, Remark, p. 9].

In the normalization $\xi = \frac{1}{2}s(s-1)\Lambda$ of Definition 1.16, and with $\rho = \frac{1}{2} + 111.029535543\ldots i$ (the 34th zero), they amount to

$$-\operatorname{Re}\{\xi'(\rho)\xi(1+\rho)\} \approx -1.34727512680\cdot 10^{-69}, \qquad \operatorname{Re}\{\xi(1+282i)/\xi(2+282i)\} \approx -0.000131957.$$

Conrey and Li normalize $\xi = s(s-1)\Lambda$, which multiplies the first value by four, giving $-5.38910050718 \cdot 10^{-69}$; there is no complex conjugation in the product. An evaluation at 65 significant digits with the `mpmath` library, recorded in the research record of this repository, reproduces both values and signs; it is a high-precision evaluation, not an interval-certified one. These witnesses refute the specific positivity conditions, not de Branges theory, not the identity of Suzuki discussed in Section 12, and not all uses of canonical systems.

It is natural to expect that the theta kernel displayed in version 1 of the zeta spectral triples paper transforms to F, and that a Gaussian tail can be dropped when trial functions are truncated; in fact the displayed kernel transforms to $F/4$, and the tail is positive.

Definition 9.31. The *displayed kernel* is $k_0(u) = \frac{\pi}{2}u^2(2\pi u^2 - 3)e^{-\pi u^2}$ for $u > 0$, and its *periodization* is the function $\mathcal{E}k_0(u) = u^{1/2}\sum_{n\geq 1} k_0(nu)$. The *Mellin transform* of k_0 is $M_{k_0}(s) = \int_0^\infty k_0(u)u^{s-1}\,du$. For $\lambda > 0$ the *truncation* is $k_{0,\lambda} = k_0 \cdot 1_{(0,\lambda]}$.

Proposition 9.32. *Let k_0, $\mathcal{E}k_0$, M_{k_0}, and $k_{0,\lambda}$ be as in Definition 9.31.*

(i) For $\operatorname{Re} s > 0$ the integral $M_{k_0}(s)$ converges absolutely and $M_{k_0}(s) = \frac{1}{8}s(s-1)\pi^{-s/2}\Gamma(s/2)$.
(ii) For $\operatorname{Re} s > 1$, $\int_0^\infty \mathcal{E}k_0(u)\,u^{s-1/2}\,\frac{du}{u}$ converges absolutely and equals $\zeta(s)M_{k_0}(s) = \frac{1}{4}\xi(s)$.
(iii) Let $\lambda \geq 1$. For every $u > 0$, $\sum_{n\geq 1} k_0(nu) - \sum_{n\geq 1} k_{0,\lambda}(nu) = \sum_{n>\lambda/u} k_0(nu) > 0$.

Proof of (i). For $\operatorname{Re}\beta > 0$, the substitution $v = \pi u^2$, under which $u = (v/\pi)^{1/2}$ and $du/u = dv/(2v)$, gives

$$\int_0^\infty u^\beta e^{-\pi u^2}\frac{du}{u} = \frac{1}{2}\int_0^\infty \Big(\frac{v}{\pi}\Big)^{\beta/2} e^{-v}\frac{dv}{v} = \frac{1}{2}\pi^{-\beta/2}\Gamma(\beta/2),$$

the integrals converging absolutely because $|u^\beta| = u^{\operatorname{Re}\beta}$ and $\operatorname{Re}\beta > 0$. Now $k_0(u)u^{s-1} = \big(\pi^2 u^{s+4} - \frac{3\pi}{2}u^{s+2}\big)e^{-\pi u^2}u^{-1}$, and $\operatorname{Re}(s+2) > 0$, so

$$M_{k_0}(s) = \frac{\pi^2}{2}\pi^{-\frac{s}{2}-2}\Gamma\big(\tfrac{s}{2}+2\big) - \frac{3\pi}{4}\pi^{-\frac{s}{2}-1}\Gamma\big(\tfrac{s}{2}+1\big) = \frac{1}{2}\pi^{-s/2}\Big[\Gamma\big(\tfrac{s}{2}+2\big) - \tfrac{3}{2}\Gamma\big(\tfrac{s}{2}+1\big)\Big].$$

Using $\Gamma(\frac{s}{2}+2) = (\frac{s}{2}+1)\frac{s}{2}\Gamma(\frac{s}{2})$ and $\Gamma(\frac{s}{2}+1) = \frac{s}{2}\Gamma(\frac{s}{2})$, the bracket is $\frac{s}{2}(\frac{s}{2}-\frac{1}{2})\Gamma(\frac{s}{2}) = \frac{1}{4}s(s-1)\Gamma(\frac{s}{2})$. □

Proof of (ii). Let $\sigma = \operatorname{Re} s > 1$. The integrand is $\sum_n k_0(nu)u^{s-1}$, and by Tonelli's theorem and the substitution $v = nu$, $\sum_n \int_0^\infty |k_0(nu)|u^{\sigma-1}du = \sum_n n^{-\sigma}\int_0^\infty |k_0(v)|v^{\sigma-1}dv < \infty$, the last integral being finite by the proof of (i). By Fubini's theorem the integral equals $\sum_n n^{-s}M_{k_0}(s) = \zeta(s)M_{k_0}(s)$, and by (i) this is $\frac{1}{8}s(s-1)\Lambda(s) = \frac{1}{4}\xi(s)$ (Definition 1.16). □

Proof of (iii). The difference is the sum over n with $nu > \lambda$. For $v > 0$, $k_0(v) > 0$ if and only if $v > (3/2\pi)^{1/2}$, and $(3/2\pi)^{1/2} < 1 \le \lambda$; so every term with $nu > \lambda$ is positive, and there are infinitely many. The series converges by the Gaussian decay of k_0. □

The kernel and its periodization are classical: summed over n, it is, up to normalization, the kernel of Riemann's second integral for ξ [Rie59, p. 4], and in the variable $u = e^t$ it is Pólya's kernel Φ [dB50, (7.2)–(7.3), p. 220]. With $s = \frac{1}{2} - iz$, (ii) gives the transform $F(z)/4$ (Definition 1.16). In [CCM25] arXiv:2511.22755v1, displays (7.1)–(7.2) present this kernel and its periodization; with the multiplicative Fourier transform used there, the transform is $F/4$ rather than F. Multiplying the kernel by four repairs the normalization and leaves the zeros unchanged. The finding concerns that version and those conventions only; no claim is made about later versions. Item (iii) shows that truncating the support of the kernel leaves a positive, nonzero tail, which a convergence argument for truncated trial functions must bound explicitly (a Gaussian bound suffices for this purpose); the detailed prolate estimate is not redeveloped here.

It is natural to expect that the printed formulas of Suzuki's comparison can be combined literally; in fact the printed even extension to negative time annihilates every even test function.

Definition 9.33. The *standard bump* is the function ψ_0 on $\mathbb{R}$ with $\psi_0(t) = \exp(-1/(1-t^2))$ for $|t| < 1$ and $\psi_0(t) = 0$ otherwise.

Proposition 9.34. *Let $\Omega \subset \mathbb{C}$ be a set with $\overline{\Omega} = \Omega$, and let $(S_t)_{t\in\mathbb{R}}$ be a family of functions $\Omega \to \mathbb{C}$ such that $S_{-t} = S_t$ for every t and $t \mapsto S_t(z)$ is locally integrable on $\mathbb{R}$ for every $z \in \Omega$. Let $S_t^{\#}$ be the reflection of S_t (Definition 3.2). Then:*

(i) for every even $\psi \in C_c^\infty(\mathbb{R})$ and every $z \in \Omega$,

$$\int_{\mathbb{R}} S_t^{\#}(z)\, i\psi'(t)\, dt = 0;$$

(ii) the standard bump ψ_0 of Definition 9.33 is a nonzero even element of $C_c^\infty(\mathbb{R})$.

Proof of (i). Fix $z \in \Omega$ and put $G(t) = S_t^{\#}(z) = \overline{S_t(\bar z)}$, which is defined since $\bar z \in \Omega$ and is locally integrable; then $G(-t) = \overline{S_{-t}(\bar z)} = G(t)$. Since ψ is even, ψ' is odd, and the substitution $t \mapsto -t$ gives $\int_{-\infty}^0 G\psi' = \int_0^\infty G(u)\psi'(-u)\,du = -\int_0^\infty G\psi'$; the two halves cancel. □

Proof of (ii). The function $t \mapsto \exp(-1/(1-t^2))$ is smooth on $(-1,1)$ and all its derivatives tend to 0 as $|t| \to 1^-$, so ψ_0 is smooth; it is supported in $[-1,1]$, even, and $\psi_0(0) = e^{-1} \neq 0$. □

Definition 9.35. In [Suz25] (version 3), with $E = E_F = F + iF'$ and $\vartheta = E^{\#}/E$, functions $S_t = \frac{i}{2}(1+\vartheta^{\#})P_t$ and P_t are defined for $t \ge 0$ by (1.5)–(1.6) on p. 3, and P_t is expressed as

a series over the zeros of F in (3.2) on p. 7 (Proposition 3.1 of the source); the source extends them to $t<0$ evenly, $S_{-t}=S_t$ after (1.6) on p. 3 and $P_{-t}=P_t$ after (3.2) on p. 7. The *literal transform* of (1.7)–(1.8) is $T_{\mathrm{lit}}\psi=\int_{\mathbb{R}} S_t^{\#}\, i\psi'(t)\,dt$ for $\psi\in C_c^\infty(\mathbb{R})$ (denoted $\widehat{P}_{D\psi}$ there), and (3.9) on p. 10 sets $\|\psi\|_0=\pi^{-1/2}\|T_{\mathrm{lit}}\psi\|_{L^2(\mathbb{R})}$, which Lemma 3.2 on p. 10 asserts to be a norm.

Remark 9.36. Let Ω be the complement in $\mathbb{C}$ of the zeros of F, E, and $E^{\#}$; then $\overline{\Omega}=\Omega$. For $z\in\Omega$ and $t\geq 0$, the printed $P_t(z)$ equals the series (3.2) of the source, $\sum_\gamma m_\gamma(e^{-i\gamma t}-1)/(\gamma(z-\gamma))$, over the distinct zeros γ of F with multiplicities m_γ. Since $|\operatorname{Im}\gamma|<\frac{1}{2}$, one has $|e^{-i\gamma t}-1|\leq 2e^{|t|/2}$, and $|z-\gamma|\geq|\gamma|/2$ for all but finitely many γ; so the terms are bounded by $4e^{|t|/2}m_\gamma|\gamma|^{-2}$ for all but finitely many γ, which is summable by (3.2). The series therefore converges uniformly for t in compact sets, and $t\mapsto S_t(z)$ is continuous on $[0,\infty)$; after the even extension it is locally integrable on $\mathbb{R}$. Applied to the printed definitions on this Ω, Proposition 9.34 gives $T_{\mathrm{lit}}\psi=0$ on Ω for every even $\psi\in C_c^\infty(\mathbb{R})$, in particular for ψ_0. Hence the literal expression (3.9) of Definition 9.35 vanishes on a nonzero test function: it is a degenerate seminorm, not the norm asserted in the literal Lemma 3.2. This uses no hypothesis on the zeros of ζ. It does not refute [Suz25, Theorem 1.1], a Hilbert-space statement proved independently of this transform. The all-real expansion (3.6)–(3.8) of the source does not use the even extension. The author has corrected the even extension in a later paper: the factor i in (1.5) is to be replaced by $-i$, and the extension to $t<0$ by $P_t(z)=\overline{P_{-t}(\bar{z})}$ [Suz26, p. 27]; this corrected definition is the reconstruction of Section 12.4. The finding above concerns the printed text of version 3 of [Suz25], which retains the even extension.

Remark 9.37. There is also an incompatibility under RH. Suppose RH holds. Take a nonnegative even $\psi\in C_c^\infty(\mathbb{R})$, not identically zero, with support so close to 0 that its cosine transform is positive at the ordinate of one zero of ζ. In the spectral expression of the Weil form Q_W (Definition 3.9), every term is then a nonnegative squared modulus and that term is positive, so $Q_W(\psi)>0$. Since $T_{\mathrm{lit}}\psi=0$, the literal transform cannot satisfy the norm identity $\|T\psi\|^2=\pi Q_W(\psi)$ at ψ. This observation assumes RH and is not a refutation of an equivalence with RH.

9.5. **Register of corrected expectations.** The following table lists the distinct expectations corrected above, each mapped to the numbered statements that correct it. Repeated mentions of one obstruction count once; positive approximation statements are not counted as refutations.

Entry	Expectation corrected	Corrected by
O01	a formal arithmetic cohomology is already a positive realization of the zeros	Prop. 6.9; Sections 5, 6
O02	skew-symmetry controls the whole operator spectrum	Prop. 9.6
O03	duality and a quaternionic star force purity	Prop. 6.11
O04	a separately chosen positive star also commutes with the flow	Prop. 6.12
O05	integer-twist cones see summands with nonreal spectrum	Prop. 9.3
O06	acyclic blocks preserve determinant and special-value data	Prop. 9.3
O07	HB (logarithmic) positivity extends downwards automatically	Prop. 9.12
O08	the HB test or a scalar realization supplies simplicity or multiplicities	Prop. 9.8

Entry	Expectation corrected	Corrected by
O09	inverse canonical-system theory proves its own positivity hypothesis	Prop. 9.12; Section 12
O10	the displayed version-1 kernel transforms to F with no factor four	Prop. 9.32
O11	a positive Fourier kernel forces real zeros	Prop. 9.14
O12	compactly supported trials allow omission of the Gaussian tail	Prop. 9.32
O13	convergence of trial transforms identifies ground eigentransforms	Prop. 9.20
O14	restricted-support positivity already covers all primes and supports	Prop. 9.16
O15	a compact error preserves positivity	Prop. 9.16
O16	shifted quotient positivity proves the original positivity; hypotheses can be dropped	Prop. 9.16
O17	L^2 convergence on growing supports implies convergence in a complex strip	Prop. 9.20
O18	a small residual identifies an eigenline without a gap bound	Prop. 9.20
O19	prolate approximation controls an unbounded nonlocal arithmetic operator	Prop. 9.20
O20	the real-coefficient no-go is either irrelevant or universal	Rem. 9.21; Prop. 9.22
O21	arbitrary smooth leafwise cohomology has harmonic representatives and admits passage of traces	Rem. 9.24
O22	the finite-field, local, and motivic results give a global realization for RH	Prop. 9.26; Rem. 9.27; Sections 10, 11
O23	positivity of heights on selected classes proves full spectral positivity	Prop. 6.9

The findings recorded in Remarks 9.28 and 9.30 and in Propositions 9.10 and 9.34 are corrections to possible uses of more recent sources and correspond to no entry of the table; Proposition 9.8 refines O08 to a statement about cyclic operators. In total, the seventeen findings R1–R17 are carried as numbered statements: fourteen of them (R1–R5, R7, R9–R12, and R14–R17) by propositions, in fifteen environments because R2 is split into Propositions 6.11 and 6.12, and three (R6, R8, and R13) by Remarks 9.24, 9.28, and 9.30; the imported part of R5 is recorded in Remark 9.21. Every entry O01–O23 of the table is corrected by at least one numbered statement.

The second table lists the distinct expectations corrected in the statements about the main theorem, its calibration, and the class of zeta functions (Sections 4, 5, and 7). Entries that sharpen an entry of the first table say so.

Entry	Expectation corrected	Corrected by
O24	all zeros of the zeta function of a higher-dimensional arithmetic scheme lie on one line	Prop. 7.27; Thm. 7.28

Entry	Expectation corrected	Corrected by
O25	the total zeta function determines the spectra in each degree, and degrees may cancel whatever the positivity	Prop. 7.30; Thm. 7.28
O26	the correct spectrum with perfect duality is already a nontrivial arithmetic statement (sharpens O01)	Props. 7.6, 7.7
O27	either half of AW4 (a positive form, or a flow-compatible one) forces purity (sharpens O03, O04)	Prop. 7.7
O28	Galois descent forces every isotypic divisor and adds content beyond zero location	Props. 7.12, 7.14
O29	a real-coefficient theory is compatible with descent (sharpens O20)	Prop. 7.15
O30	a cohomology for a single analytic L-function carries content beyond its Riemann hypothesis	Prop. 7.6
O31	finite-dimensional Weil formalism with duality, rational zeta function, functional equation, and integral counts forces purity	Prop. 4.6
O32	a nondegenerate Frobenius-compatible form suffices	Prop. 4.6
O33	a nondegenerate Frobenius-compatible pairing on the algebraic classes of X itself controls H^j	Prop. 4.6
O34	Weil's positivity is a canonical positive Frobenius polarization on H^1	Props. 4.11, 4.12
O35	the suspension may use one logarithm of each eigenvalue	Prop. 7.22
O36	the determinant of suspended Frobenius is known only up to an unspecified constant	Thm. 7.20; Prop. 7.21
O37	the finite-field zeta function satisfies the axioms verbatim, with the arithmetic unit	Prop. 7.23; Thm. 7.24
O38	Deligne's theorem makes the positivity axiom a theorem for every smooth projective variety over $\mathbb{F}_q$	Thm. 7.24(iv); Rem. 7.25
O39	the regularized determinant over the zeros of ζ is $\xi/2\pi^2$, with normalization term one	Thm. 7.32; Thm. 7.37 (O57 for general K)

Every entry O24–O38 is corrected by a numbered proposition or theorem proved here or imported with a locator (the carrier of O34 uses the imported semisimplicity and purity statements), and O39 by an imported theorem; its analogue for general number fields is O57 below. The findings recorded in Propositions 4.4, 9.4, and 9.1 are further corrected expectations of the same kind.

The third table lists further corrected expectations, about the multiplicative form of Weil's positivity, the parity of central orders, and the Künneth conjecture of Section 8.

Entry	Expectation corrected	Corrected by
O40	Weil's Rosati positivity is literally an instance of the additive algebra form, and the positivity axiom on the suspension of a curve can only be derived from the Riemann hypothesis for the curve	Props. 4.19, 4.20; Thm. 7.24(vi)

Entry	Expectation corrected	Corrected by
O41	self-duality and a simple pole at $s = 1$ suffice for the collapse of a cohomology for one L-function (sharpens O30)	Prop. 7.9
O42	a Künneth formula holds for fibre products over $\operatorname{Spec}\mathbb{Z}$	Prop. 8.5
O43	a derived Künneth formula over a base holds without a hypothesis on the map (it holds, for every cochain enhancement, when the derived pushforward of the constant sheaf lies in the thick subcategory generated by the constant sheaf, and fails for the double cover $E \to \mathbb{P}^1$ over the cochains as over the cohomology ring)	Props. 8.6, 8.9, 8.27, 8.30; Rem. 8.10
O44	the tensor product of the cohomologies of two arithmetic curves is a spectral core, on which AW2 and AW-det make sense	Thm. 8.12
O45	the sums $\rho + \rho'$ of zeros form a discrete spectrum	Thm. 8.12(iii) (under a hypothesis on the zeros)
O46	Kurokawa's cone is the cohomology of the arithmetic surface	Prop. 8.13
O47	compact resolvent (AW6) survives tensor products, and geometric compactness repairs the product	Props. 8.15, 8.16
O48	the suspension of a product over $\mathbb{F}_q$ is the tensor product over $\mathbb{C}$ of the suspensions	Prop. 8.17; Thm. 8.23
O49	the absolute product of closed orbits is a well-behaved spectral object	Prop. 8.18
O50	the trace formula of the product is a Lefschetz formula over isolated closed orbits	Prop. 8.19
O51	archimedean factors of tensor products are obtained by adding spectra	Prop. 8.20
O52	the real-coefficient obstruction is an accident of degree one (sharpens O20, O29)	Rem. 8.21
O53	positivity on the arithmetic surface is a condition unrelated to positivity on the curve, and the bare Künneth conjecture carries content beyond the Riemann hypothesis	Thm. 8.34; Prop. 8.37
O54	for a given star, the Hodge index clause on the arithmetic surface is equivalent to the algebra form AW4-E of positivity	Prop. 8.36
O55	the divisor identity for an open object decides the connecting maps of its localization sequence	Prop. 8.22
O56	the counting bound for a tensor sum of semibounded spectra holds with the shifts c_2 and c_1, or with $c_1 + c_2$ for arbitrary signs	Prop. 8.15(iii), (v)

Every entry O40–O56 except O52 is corrected by a numbered statement proved here; O52 is carried by a remark with an imported statement, and O45 by a statement proved under a stated hypothesis on the zeros.

The fourth table lists the corrected expectations about the normalization of the determinant over a number field (Section 7.7) and about the trace-formula axiom (Section 7.8).

Entry	Expectation corrected	Corrected by
O57	for every number field the regularized determinant of the zeros reproduces Λ_K with normalization term one (sharpens O39)	Thm. 7.37
O58	zeta-regularizability of $\operatorname{div}\xi_K$ follows from its counting function, main term, and error $O(\log T)$	Prop. 7.40
O59	the normalization over a number field cannot be determined because the half-plane splitting of the spectral zeta function needs a finer regularization	Thm. 7.37; Prop. 7.39
O60	the half-plane pieces $Z_\pm$ of the spectral zeta function are regular at $w = 0$	Prop. 7.39
O61	for the zeros of ζ_K the $O(\log T)$ fluctuation of the count contributes to the residues of $Z_\pm$ at $w = 0$	Prop. 7.39 (for general spectra it can: Prop. 7.40)
O62	the determinant of $\operatorname{div}\xi_K$ requires the general theory of regularized products	Thm. 7.37
O63	the trace formula on a bounded window $(0, L)$ of times determines the degree-one spectrum	Thm. 7.50
O64	finitely many Lefschetz identities determine the characteristic polynomial of Frobenius	Prop. 7.51
O65	a two-sided window $(-L, L)$ gives no more than the one-sided window $(0, L)$	Prop. 7.52
O66	a faithful trace-formula axiom must be one-sided, the two-sided form being stronger than Lefschetz	Thms. 7.45, 7.43
O67	AW5 is a purely analytic condition, so that AW5 with AW6′ implying AW2 is the analogue of "Lefschetz implies rationality"	Prop. 7.46; Thm. 7.47
O68	a class of test functions determines the spectrum if and only if it separates points	Thm. 7.50; Prop. 7.42
O69	the trace formula integrates to the determinant identity, as over $\mathbb{F}_q$	Thms. 7.43, 7.37
O70	the clause of AW6 that the completion loses no spectrum is independent of AW5	Prop. 7.46
O71	an exponential sum over a discrete spectrum determines its coefficients only if it vanishes on all positive times	Prop. 7.42(ii); Thm. 7.50
O72	Schwartz functions of the spectral variable are admissible test functions	Prop. 7.53
O73	replacing AW2 by a trace-formula axiom weakens the arithmetic axioms, as the Lefschetz formula makes rationality a theorem	Thm. 7.43
O74	over the étale cochain algebras the derived Künneth formula for $E \times_{\mathbb{P}^1} E$ is repaired, the failure over the cohomology ring being due to the coefficient algebra	Props. 8.8, 8.9

Entry	Expectation corrected	Corrected by
O75	the failure of the derived Künneth formula for $E \times_{\mathbb{P}^1} E$ is a defect of the base $\mathbb{P}^1$	Props. 8.29, 8.30

Every entry O57–O75 is corrected by a numbered statement proved here. The following seven expectations arose in the comparison with BSD in Subsection 2.2. The examples concern stated formal deductions, not elliptic curves violating BSD.

Entry	Expectation corrected	Corrected by
O76	the geometric divisor class of E counts its rational points	Prop. 2.1
O77	a positive height or spectral form makes the central Abel–Jacobi map onto	Prop. 2.3
O78	a spectral divisor and positive metric fix the special-value coefficient and regulator lattice	Prop. 2.7; Prop. 7.30 for alternating cancellation
O79	determinant order always equals central eigenspace dimension	Prop. 2.4
O80	duality and purity alone supply the needed positive form	Prop. 4.6(ii)–(iii); Rem. 7.25
O81	finiteness of every prime-primary subgroup proves finiteness of the whole group	Prop. 2.8
O82	failure of a direct implication proves logical nonimplication of all geometric standard conjectures and BSD	Rem. 2.2

The statement that a trace formula determines only the alternating divisor of a higher-dimensional object is entry O25; for the trace formula see the paragraph after Theorem 7.43.

10. Candidate theories

The two constructions below test different parts of the same proposed theory. Flach–Morin cohomology tests its integer twists; CCM adelic cyclic homology tests the middle zero spectrum over a number field. Neither comparison is an additional cohomology axiom.

Definition 10.1 (Comparison probes)**.** Let $\overline{X} \in \overline{\mathrm{Sch}}^{\mathrm{proj}}$ be connected, let $D_X = H(\overline{X})$ for a Deninger cohomology theory H (Definition 1.21), and let K be a number field.

(a) Put $R = \mathbb{C}[u]$. A *derived enhancement* of D_X is a complex $D(X)$ of R-modules whose cohomology is $H^\bullet(X)$, with u acting as Θ. Write $D_c(X)$ for a compact-support enhancement when the input includes support data.

(b) For $n \in \mathbb{Z}$ the *integer-fibre probe* is

$$P_n(E) = R\,\mathrm{Hom}_R(R/(u-n), E).$$

Its connecting endomorphism of degree one is denoted β_n.

(c) Put $C_K = \mathbb{A}_K^\times / K^\times$. Let $N_K = H^1_{\mathrm{ad}}(K)$ be CCM's relative cyclic-homology representation, formed using the quotient by the closure of the restriction map's range (Definition 11.18). Its *trivial norm-one sector* is $N_{K,1} = N_K^{C_{K,1}}$, where $C_{K,1} = \ker(|\cdot| : C_K \to \mathbb{R}_+^\times)$. Let T_K be the infinitesimal generator of the induced $C_K/C_{K,1} \simeq \mathbb{R}_+^\times$ action ϑ_m on smooth vectors, parameterized by norm e^t.

(d) If the continuous dual of $N_{K,1}$ has finite-dimensional generalized eigenspaces $E_\rho = \bigcup_{r\geq 1}\ker(T'_K-\rho)^r$ with discrete support, where T'_K is the transpose, its *semisimple spectral probe* is

$$\mathrm{Core}^{\mathrm{ss}}(N_{K,1}) = \bigoplus_\rho E_\rho^\vee, \qquad \Theta|_{E_\rho^\vee} = \rho\,\mathrm{id}.$$

The two-term free resolution of $R/(u-n)$ gives $P_n(E)\simeq \mathrm{Fib}(\Theta-n : E\to E)$. When E lies in degree zero, its two terms lie in degrees zero and one. The fibre is taken in complexes of $\mathbb{C}$-vector spaces: over $\mathbb{F}_q$, $\Theta-n$ need not be $\mathbb{1}_q$-linear. The sector $N_{K,1}$ is not fixed by all of C_K. With the Mellin convention of [CCM09, Theorem 4.16 and (4.41)], T_K has the unshifted zero parameter ρ; the older half-density action W has parameter $\rho-1/2$. The finite dual retains $\dim E_\rho$, while the displayed action discards only nilpotent parts. The finite-block and trace properties of this extraction remain to be established for the CCM topology.

Conjecture 10.2 (Common realization of two probes)**.** *The single assignment of Conjectures 1.26 and 1.28 admits a multiplicative derived enhancement, with compact-support objects and comparison maps, such that the following hold.*

(i) For each connected $\overline{X}\in\overline{\mathrm{Sch}}^{\mathrm{proj}}$ covered by the hypotheses of [FM20, Conjectures 2.3 and 4.1, and Remark 4.15]*, the Flach–Morin compact-support complex $R\Gamma_{\mathrm{ar},c}(X,\widetilde{\mathbb{C}}(n))$ at each $n\in\mathbb{Z}$ is naturally equivalent to $P_n(D_c(X))$. Likewise without supports. The equivalences carry cup product with θ to β_n; over $\mathbb{F}_q$ this operator corresponds to $\log(q)\cup e_{\mathbb{F}_q}$, with the generator convention of* [FM20, Proposition 3.8].

(ii) For each number field K, the extraction $\mathrm{Core}^{\mathrm{ss}}(N_{K,1})$ is defined and naturally isomorphic, with its generator and spectral multiplicities, to $H^1D(\overline{X}_K)$. This comparison matches the integrated norm-action traces for $C_{K,1}$-invariant functions in $\mathbf{S}(C_K)$, as in [CCM09, Theorem 4.16]*. It is natural under field isomorphisms. A pairing on the extracted sector, still to be constructed, is carried to the pairing $H^1_\rho\times H^1_{1-\rho}\to\mathbb{C}$ of AW3.*

This is a conjecture of these notes about comparison maps, not a conjecture attributed in full to either source. The first clause refines Flach–Morin's proposed long exact sequence and derived-fibre remark; the second asks for structure beyond CCM's proved spectral trace formula. It does not identify the raw cyclic module with the derived cohomology complex or equate the CCM trace form with the star form of AW4. Extension maps between fields, integral determinant lattices, and the nontrivial Hecke-character sectors require additional constructions.

For $X=\mathrm{Spec}\,\mathbb{F}_q$ the unit $\mathbb{1}_q$ gives a quick check. Its kth Fourier mode has eigenvalue $2\pi ik/\log q$. Thus P_0 has one copy of $\mathbb{C}$ in each of degrees zero and one, while P_n is acyclic for every nonzero integer n. This is the finite-field fixed-twist pattern of Theorem 11.16, subject to its finite-generation hypothesis and cup-product normalization. CCM's global-field construction does not apply to this zero-dimensional point.

The two probes are incomplete in different ways. The family of integer fibres alone cannot recover noninteger eigenvalues (Proposition 9.3); the CCM object here reaches degree one only. A scalar positive trace-form completion can lose spectral multiplicity (Proposition 9.8), which is why the proposed CCM extraction keeps the entire generalized block before semisimplifying. After the proposed identification, AW4 forces the target generator to be semisimple (Theorem 1.15). The table records the proved portions and the remaining obligations.

Table 8 separates constructed cohomology theories from conjectures about their coefficients, comparison maps, or positivity. The final column concerns the global arithmetic realization of Conjectures 1.26 and 1.28. It records what does not follow from the cited results; it is not a claim that no other approach could supply the required construction. Section 11 states the comparison theorems with their hypotheses.

Table 8: Existing theories and their arithmetic obligations.

Framework	Constructed objects or proved results	Remaining requirement for the arithmetic realization
Deninger's dynamical cohomology	The formal implication from a positive compatible star to purity; geometric dynamical analogues under their stated hypotheses [Den10, pp. 4–5]	A global arithmetic cohomology with the full zeta spectrum, duality, and positive star remains conjectural [FM20, Conj. 2.3].
Flach–Morin cohomology (Weil–Arakelov)	Real complexes for proper regular arithmetic schemes are defined [FM20, §3]	Their fixed-twist comparison with dynamical cohomology is Conjecture 4.1; Theorem 4.3 proves it over finite fields under finite generation, without supplying the positive star.
Ayoub's new Weil cohomology	A theory on smooth residue-field varieties over a common coefficient algebra A_K, with rigid and ℓ-adic specializations [Ayo20, Thm. 1.5]	The domain property and integration injectivity are Conjectures 11.10 and 11.11; no global arithmetic flow or positive form is supplied.
Ayoub's foliated topology	The Poincaré lemma and de Rham comparison for differential modules [Ayo19, Thm. 5.7.4, Cor. 5.7.7]	This topology concerns differential equations. A comparison with arithmetic dynamical cohomology is additional; the separate conservativity proof has the author's recorded gap [Ayo, conservativity manuscript notice].
Connes–Consani archimedean cyclic homology	Archimedean cyclic homology realizes Serre's local gamma factors by regularized determinants [CC14b, Thm. 1.1]	A global zero-spectrum identification and compatible positive form do not follow from this local formula.
CCM adelic cyclic homology	The relative H^1 of the cyclic-module restriction-map cokernel realizes Hecke L-function zero spectra [CCM09, Definition 4.10 and Proposition 4.13, pp. 21–23]	A full graded theory on arithmetic schemes, and a positive star, are not supplied; the adelic trace-form positivity is RH-equivalent [CCM09, Corollary 6.3, p. 29].
Arithmetic and scaling sites	Geometric sites and Riemann–Roch on periodic orbits [CC17, Thms. 3.9, 5.17]; Riemann–Roch for arithmetic divisors [CC23, Thm. 1.1]	The required positivity on the square remains a target of the Riemann–Roch strategy [CC19, §3]; the one-dimensional formulas do not prove it.
Topological periodic cyclic homology (TP)	Hesselholt's finite-field determinant formula for the full even and odd sums [Hes18, Thm. A]	A global arithmetic object, comparison across places, and positive global form are additional requirements.
Prismatic and A_{inf} cohomology	Local integral comparison theorems [BMS18, Thm. 1.8], [BS22, Thm. 1.8]	These comparisons do not construct a complex global flow whose spectrum is the zeta divisor.
Kottwitz and Berkovich settings	The Kottwitz-valued Weil theory is conjectural [Sch18, Conj. 9.5]; Berkovich motives have pointwise comparison theorems [Sch26a, Prop. 1.5]	Neither statement supplies the global zero spectrum or its positive completion.

The distinction is between these existing constructions and the additional arithmetic realization and positivity assertions, not between having or lacking cohomology in every sense. In

particular, Deninger's real-coefficient obstruction concerns the specified functorial Galois multiplicities [Den23, Thm. 2.13]; it does not contradict the existence of the real Weil–Arakelov complexes in the table.

Several pieces of the dynamical cohomology $H^\bullet_{\mathrm{dyn}}$ of [FM20, Conj. 2.3] already have rigorous, unconditional constructions; Table 9 lists them. None supplies the positivity that would prove RH. In each case where a positive form is available, its existence is equivalent to RH.

Table 9: Rigorous partial realizations of dynamical cohomology.

Piece of $H^\bullet_{\mathrm{dyn}}$	Rigorous construction	What it does not supply
Smooth projective varieties over $\mathbb{F}_q$	Deninger's spaces $D(H^i(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) \otimes \mathbb{C})$ with their derivation Θ [Den92], [FM20, §4.2, (7)]; when Frobenius acts semisimply they satisfy [FM20, Conj. 2.3] except the $*$-operator (iv)	The positive $*$-operator; positivity here is the Riemann hypothesis over $\mathbb{F}_q$, which is imported, not produced.
Degree one for $\overline{\operatorname{Spec}\mathbb{Z}}$	The relative H^1 of the adelic cyclic module realizes Hecke L-function zero spectra [CCM09, Def. 4.10, Prop. 4.13]; an operator on a nuclear Fréchet space has the zeros of ξ as eigenvalues, with algebraic multiplicity equal to the zero order [Mey05, Thm. 4.1, Cor. 4.2]	Products, duality, a star, and the other degrees. Positivity of the adelic trace form is equivalent to RH [CCM09, Cor. 6.3]. A zero of order $m > 1$ gives a Jordan block of length m, which a positive equivariant form excludes, so positivity would also force simple zeros.
The parts $\Theta = n$, $n \in \mathbb{Z}$	Weil–Arakelov complexes, defined unconditionally from motivic and Deligne cohomology [FM20, §3]; their comparison with $H^\bullet_{\mathrm{dyn}}$ is [FM20, Conj. 4.1], proved over $\mathbb{F}_q$ under finite generation [FM20, Thm. 4.3]	Noninteger eigenvalues, and the comparison over $\mathbb{Z}$.
Archimedean factor	Archimedean cyclic homology realizes Serre's local gamma factors by regularized determinants [CC14b, Thm. 1.1]	The nontrivial zeros.
The full theory on $\overline{\operatorname{Spec}\mathbb{Z}}$	None: cohomology in every degree with cup product, duality and a positive $*$-operator is conjectural [FM20, Conj. 2.3]	Everything listed in the other rows at once.

The tautological constructions show the same circularity. The trivial model of Definition 7.5 exists for every divisor, but a positive datum with that spectrum exists exactly when the divisor lies on $\operatorname{Re} s = \frac{1}{2}$ (Proposition 7.6, (iii)); the Lean statements recorded in Subsection 14.13 check this equivalence. A rigorous candidate in the useful sense would be a full theory constructed without reference to the zeros, with positivity proved rather than imposed.

Scholze's real-exponent analytic ring offers another possible input. For $0 < r < 1$, let $R_{>r} = \mathbb{Z}((T^{\mathbb{R}}))_{>r}$ be the union over $r < r' < 1$ of series $\sum_x a_x T^x$ with $\sum_x |a_x|(r')^x < \infty$. Lecture XIV of [Sch26b, Theorem 14.11] equips it with an analytic ring structure and the semigroup action $\sigma_t(T^x) = T^{tx}$ for $t \geq 1$. At primes these are Frobenius lifts. A fixed-radius action need not be invertible.

Proposition 10.3. *On $R_{>1/4}$, the endomorphism σ_2 is not surjective.*

Proof. The series $b = \sum_{n\geq 0} 3^n T^n$ belongs to $R_{>1/4}$: its weighted sum converges whenever $1/4 < r' < 1/3$. A preimage under σ_2 must be $\sum_{n\geq 0} 3^n T^{n/2}$, whose weighted sum $\sum_{n\geq 0}(3\sqrt{r'})^n$ diverges for every $r' > 1/4$. Coefficient uniqueness rules out another preimage. □

The source also gives canonical isomorphisms between rings at different radii. The proposition only blocks treating this literal fixed-radius semigroup as the two-sided flow required above; a radius-varying or enlarged construction needs its own comparison, generator, and trace.

No composition of the comparison arrows of Theorems 11.3, 11.5, 11.6, 11.7, 11.8, 11.9, 11.12, and 11.14 produces the object of Theorem 6.3: their domains do not match the proposed arithmetic spectral source without new constructions (Remark 9.27; cf. Proposition 9.26).

11. Imported results

This section states, with their hypotheses and exact locators, the results from the literature on which Sections 9, 10, and 12 rely. They are not proved here; the standing convention of these notes is that advanced comparison theorems are stated precisely and not re-proved. Page numbers refer to the version cited in the bibliography. None of these results constructs the object of Theorem 6.3.

11.1. **Coefficients and leafwise cohomology.** Let k be a number field. Deninger lists properties 2.1–2.12 expected of a cohomology $H^\bullet(\operatorname{Spec}\overline{\mathcal{O}}_k, \mathbb{C})$ with an endomorphism θ [Den23, pp. 2–5]; property 2.12 asks that, for $k/\mathbb{Q}$ Galois with group G, each irreducible representation of G occur in the central generalized eigenspace of θ on H^1 with multiplicity equal to the order of its Artin L-function at $s = \frac{1}{2}$.

Theorem 11.1 ([Den23, Theorem 2.13, p. 5])**.** *Let k be a number field. There is no functorial real-valued cohomology theory, written $H^\bullet(\operatorname{Spec}\overline{\mathcal{O}}_k, \mathbb{R})$, with an endomorphism $\theta_{\mathbb{R}}$ such that $H^\bullet(\operatorname{Spec}\overline{\mathcal{O}}_k, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ satisfies property 2.12 of* [Den23].

The source then observes [Den23, pp. 5–6] that complex leafwise cohomology always has the real structure given by the cohomology of real forms, so that "there has to be a twisted version of leafwise cohomology with complex coefficients that does not have a natural real structure".

Reduced leafwise cohomology $\overline{H}^j_{\mathcal{F}}$ and leafwise harmonic forms are as in Definition 9.23.

Theorem 11.2 ([DS01, Proposition 1.2 and (12), p. 214])**.** *There is a one-dimensional foliation $\mathcal{F}$ with dense leaves on a three-dimensional Heisenberg nilmanifold, with a leafwise metric, such that the reduced smooth leafwise cohomology $\overline{H}^1_{\mathcal{F}}$ is infinite-dimensional while the space of smooth leafwise harmonic 1-forms is one-dimensional.*

In the same example the passage from the dynamical side of a trace formula to reduced leafwise cohomology is not allowed [DS01, §2, pp. 215–217]. The positive theorem quoted there as Theorem 0.2 [DS01, p. 210], due to Álvarez López and Kordyukov, assumes a Riemannian foliation with a bundle-like metric.

11.2. **Finite fields and p-adic comparisons.** Deligne's theorem is stated in Section 3.1 (Theorem 3.8).

Theorem 11.3 ([Hes18, Theorem A, p. 2])**.** *Let $k = \mathbb{F}_q$, $W = W(k)$, $\iota : W \to \mathbb{C}$ a ring homomorphism, and $f : X \to \operatorname{Spec} k$ smooth and proper. Let Θ be the graded derivation of periodic topological cyclic homology $\mathrm{TP}_*(X) \otimes_{W,\iota} \mathbb{C}$ with $q^\Theta = \mathrm{Fr}_q^*$ normalized as in the source. Then*

$$\zeta(X, s) = \frac{\det_\infty\big(s\cdot\mathrm{id} - \Theta \mid \mathrm{TP}_{\mathrm{odd}}(X) \otimes_{W,\iota} \mathbb{C}\big)}{\det_\infty\big(s\cdot\mathrm{id} - \Theta \mid \mathrm{TP}_{\mathrm{even}}(X) \otimes_{W,\iota} \mathbb{C}\big)}.$$

It is natural to expect Theorem 11.3 to identify each cohomological degree with a weight-graded piece carrying a metric; in fact it is an identity of regularized determinants over the full even and odd graded sums.

Remark 11.4. Theorem 11.3 is proved by comparing spectral sequences [Hes18, pp. 2–3]; it identifies regularized determinants over the full even and odd graded sums, not individual weight-graded pieces with metrics, and the choice $\iota : W \to \mathbb{C}$ is algebraic, not a continuous embedding supplying a positive metric. It concerns one finite field and says nothing about $\operatorname{Spec}\mathbb{Z}$.

Hesselholt's formula uses TP, the topological analogue of periodic cyclic homology formed as the circle Tate construction on THH, rather than TC [Hes18, Theorem A and p. 1]. The TC^- and TP filtrations of Theorem 11.6 provide p-adic comparison data. Connes–Consani's archimedean determinant instead uses ordinary cyclic homology [CC14b, Theorem 1.1]. These parallel determinant constructions do not by themselves identify either theory with CCM's adelic cyclic quotient.

Theorem 11.5 ([BMS18, Theorem 1.8, pp. 5–6])**.** *Let $C/\mathbb{Q}_p$ be complete and algebraically closed with ring of integers $\mathcal{O}_C$ and residue field k, let $\mathfrak{X}/\mathcal{O}_C$ be a proper smooth formal scheme, $A_{\mathrm{inf}} = W(\mathcal{O}_C^{\flat})$, and $\mu = [\epsilon]-1$ for a compatible system of p-power roots of unity. There is a perfect A_{inf}-complex $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})$ with a Frobenius and canonical quasi-isomorphisms $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes^L_{A_{\mathrm{inf}}} W(k) \simeq R\Gamma_{\mathrm{crys}}(\mathfrak{X}_k/W(k))$, $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes^L_{A_{\mathrm{inf}},\theta} \mathcal{O}_C \simeq R\Gamma_{\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C)$, and $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes_{A_{\mathrm{inf}}} A_{\mathrm{inf}}[1/\mu] \simeq R\Gamma_{\acute{e}t}(\mathfrak{X}_C, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}[1/\mu]$.*

Theorem 11.6 ([BMS19, Theorems 1.8, 1.10, and 1.12(4), pp. 5–7])**.** *If A is the p-adic completion of a smooth $\mathcal{O}_C$-algebra, there is a Frobenius-equivariant identification*

$$A\Omega_A \simeq R\Gamma_{\mathrm{syn}}\big(A, \pi_0 \mathrm{TC}^-(-;\mathbb{Z}_p)\big);$$

if A is smooth over a perfect field k of characteristic p, the right-hand side computes the crystalline cohomology $R\Gamma_{\mathrm{crys}}(A/W(k))$. If A is quasisyntomic, $\mathrm{TC}^-(A;\mathbb{Z}_p)$ and $\mathrm{TP}(A;\mathbb{Z}_p)$ carry complete exhaustive multiplicative filtrations with

$$\mathrm{gr}^n \mathrm{TC}^-(A;\mathbb{Z}_p) \simeq N^{\geq n}\widehat{\Delta}_A\{n\}[2n], \qquad \mathrm{gr}^n \mathrm{TP}(A;\mathbb{Z}_p) \simeq \widehat{\Delta}_A\{n\}[2n].$$

Theorem 11.7 ([BS22, Theorem 1.8, p. 4])**.** *Let (A, I) be a bounded prism and $\mathfrak{X}$ a smooth p-adic formal scheme over A/I, with prismatic complex $R\Gamma_{\Delta}(\mathfrak{X}/A)$. There are canonical comparisons with crystalline cohomology when $I = (p)$, with de Rham cohomology of $\mathfrak{X}/(A/I)$ after Frobenius base change, with Hodge–Tate cohomology for affine $\mathfrak{X}$, and, for perfect A, with étale cohomology: $R\Gamma_{\acute{e}t}(\mathfrak{X}_\eta, \mathbb{Z}/p^n) \simeq \big(R\Gamma_{\Delta}(\mathfrak{X}/A)/p^n[1/I]\big)^{\varphi=1}$, where the fixed points are derived.*

The Breuil–Kisin and A_{inf} cases are specializations [BS22, Example 1.9, p. 5]. The tensor products and fixed points in Theorems 11.5–11.7 are derived, and an associated-graded comparison is not a splitting.

Theorem 11.8 ([CC14b, Theorem 1.1 and (2), p. 2; (4), p. 3])**.** *Let K be a number field, X/K smooth projective of dimension d, and ν an archimedean place. With the archimedean cyclic homology $HC^{\mathrm{ar}}_*(X_\nu)$ and the operator Θ defined in the source through the λ-operations, one has, for real s away from the singularities,*

$$\prod_{w=0}^{2d} L_\nu(H^w(X), s)^{(-1)^{w+1}} = \frac{\det_\infty\big(\frac{1}{2\pi}(s-\Theta) \mid HC^{\mathrm{ar}}_{\mathrm{even}}(X_\nu)\big)}{\det_\infty\big(\frac{1}{2\pi}(s-\Theta) \mid HC^{\mathrm{ar}}_{\mathrm{odd}}(X_\nu)\big)}.$$

The orientation (even over odd) is the one printed and is opposite to that of Theorem 11.3. At a complex place, HC^{ar} is the cokernel in an exact sequence $0 \to HP^{\mathrm{real},\lambda}_{*+2}(X_\nu) \to HC_*(X_\nu) \to HC^{\mathrm{ar}}_*(X_\nu) \to 0$ [CC14b, (4), p. 3]. The formula realizes local gamma factors, not global zeros.

Theorem 11.9 ([Ayo20, Construction 1.3, Théorème 1.5, pp. 1749–1750])**.** *Let K be a valued field of rank one and mixed characteristic $(0,p)$ with residue field k and completion $\widehat{K}$. There is a Weil cohomology $\Gamma_{\mathrm{new}}(-/K)$ on smooth k-schemes with coefficients in a K-algebra A_K, with $\Gamma_{\mathrm{new}} \otimes^{L}_{A_K} \widehat{K} \simeq \Gamma_{\mathrm{rig}}$ and, after choosing $K \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$ for $\ell \neq p$, $\Gamma_{\mathrm{new}} \otimes^{L}_{A_K} \overline{\mathbb{Q}}_\ell \simeq \Gamma_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$.*

For a rank-one valued field K of mixed characteristic $(0,p)$, write A_K for Ayoub's algebra of abstract p-adic periods and $\widehat{K}$ for the completion of K. Write $I_K : A_K \to \widehat{K}$ for the canonical K-algebra homomorphism defined by integration in [Ayo20, Construction 1.3, (1-1)–(1-2), pp. 1749–1750].

Conjecture 11.10 (Ayoub; [Ayo20, Remarque 1.6, p. 1750])**.** *Let K be a rank-one valued field of mixed characteristic $(0,p)$ and let A_K be its algebra of abstract p-adic periods. Then A_K is an integral domain.*

Conjecture 11.11 (Ayoub; [Ayo20, Remarque 1.6, p. 1750])**.** *Let K be a number field equipped with a rank-one nonarchimedean valuation of residue characteristic p, let $\widehat{K}$ be its completion, and let A_K be its algebra of abstract p-adic periods. Then the integration homomorphism $I_K : A_K \to \widehat{K}$ is injective.*

Theorem 11.12 ([Ayo19, Théorème 5.7.4 and Corollaire 5.7.7, pp. 187–188])**.** *Let (K,Δ) be a differential field of characteristic zero.*

(i) (Théorème 5.7.4.) For an ind-holonomic differential module M, the inclusion of horizontal sections into the de Rham complex of M is an fttf*-local equivalence.*

(ii) (Corollaire 5.7.7.) For a bounded-below complex $M^\bullet$ of ind-holonomic differential modules, $H^i_{\mathrm{dR}}(K; M^\bullet) \simeq H^i_{\mathrm{fttf}}(K; (M^\bullet)^\delta)$.

The source is an author manuscript; its "foliated" topology concerns algebraic differential equations, not a foliation on an arithmetic dynamical space.

Conjecture 11.13 ([Sch18, Conjecture 9.5, pp. 24–25])**.** *There is a Weil cohomology theory for varieties over $\overline{\mathbb{F}}_p$ with values in Kottwitz's category $\mathrm{Kt}_{\mathbb{Q}}$ whose images in $\mathrm{Kt}_{\mathbb{Q}_p}$, $\mathrm{Kt}_{\mathbb{Q}_\ell}$, and $\mathrm{Kt}_{\mathbb{R}}$ are crystalline cohomology, ℓ-adic cohomology, and a Weil cohomology with values in $\mathrm{Kt}_{\mathbb{R}}$, respectively.*

The source explains that the conjecture follows from the standard conjectures, the Tate conjecture, and the Hodge conjecture for CM abelian varieties; it is not an unconditional construction, and its base is $\overline{\mathbb{F}}_p$, not $\operatorname{Spec}\mathbb{Z}$.

Theorem 11.14 ([Sch26a, Proposition 1.5, p. 4])**.** *Let C be an algebraically closed nondiscrete Banach field and $X = M_{\mathrm{arc}}(C)$. If $C = \mathbb{C}$, then $D(\mathbb{Z}) \simeq D^{\mathrm{eff}}_{\mathrm{mot}}(X)$; if C is nonarchimedean and n is invertible in its residue field, then $D(\mathbb{Z}/n) \simeq D^{\mathrm{eff}}_{\mathrm{mot}}(X, \mathbb{Z}/n)$; if the residue characteristic is p, then p is invertible in $D^{\mathrm{eff}}_{\mathrm{mot}}(X)$; and in mixed characteristic, tilting gives $D^{\mathrm{eff}}_{\mathrm{mot}}(C) \simeq D^{\mathrm{eff}}_{\mathrm{mot}}(C^\flat)$.*

11.3. Flach–Morin cohomology (Weil–Arakelov).

Conjecture 11.15 ([FM20, Conjecture 4.1, p. 527])**.** *Let X be as in the source and $n \in \mathbb{Z}$. There is a long exact sequence*

$$\cdots \to H^i_{\mathrm{ar},c}(X, \widetilde{\mathbb{C}}(n)) \to H^i_{\mathrm{dyn},c}(X) \xrightarrow{\Theta - n} H^i_{\mathrm{dyn},c}(X) \to H^{i+1}_{\mathrm{ar},c}(X, \widetilde{\mathbb{C}}(n)) \to \cdots$$

whose composite $H^i_{\mathrm{ar},c} \to H^{i+1}_{\mathrm{ar},c}$ is cup product with the class θ of the source, and likewise without supports.

Theorem 11.16 ([FM20, Theorem 4.3, p. 527])**.** *Let X be a smooth projective connected variety over $\mathbb{F}_q$ and $n \in \mathbb{Z}$. If $H^i(X_W, \mathbb{Z}(n))$ is finitely generated for all $i \in \mathbb{Z}$, then Conjecture 11.15 holds for X and n, with the dynamical cohomology constructed from ℓ-adic cohomology.*

It is natural to read Theorem 11.16 as a finite-field verification of the whole dynamical package; in fact it is the fixed-twist long exact sequence only.

Remark 11.17. Theorem 11.16 gives the fixed-twist long exact sequence of Conjecture 11.15 only. It does not supply the positive star, and the comparison of cup-product operators carries a normalization: [FM20, Proposition 3.8, p. 526] relates the Weil–Arakelov operator to $\log(q)\cup e_{\mathbb{F}_q}$. A derived enhancement $R\Gamma_{\mathrm{ar},c} \simeq \mathrm{fib}(\Theta - n)$ is a further statement. Deninger's determinant formula [Den10, Conjecture 2, p. 3] and the broader conjecture [FM20, Conjecture 2.3, p. 522] remain conjectures.

11.4. **CCM adelic cyclic homology.**

Definition 11.18 (Adelic trace form)**.** Let K be a global field and $C_K = \mathbb{A}_K^\times/K^\times$. Let $\mathbf{S}(C_K)$ be the convolution test algebra of [CCM09, (4.20), p. 19]: for a number field it is $\bigcap_{\beta\in\mathbb{R}} |\cdot|^\beta \mathcal{S}(C_K)$, and for a function field it consists of compactly supported Schwartz functions. Let $H^1_{\mathrm{ad}}(K)$ denote the source's $H^1(\mathbb{A}_K/K^\times, C_K)$, obtained by applying cyclic homology to the restriction-map cokernel. The cokernel is taken modulo the closure of the range [CCM09, Definition 4.10 and (4.28), pp. 21–22]. Write ϑ_m for its multiplicative action. For $f, g \in \mathbf{S}(C_K)$ set

$$\langle f, g\rangle_{\mathrm{ad}} = \mathrm{Tr}\big(\vartheta_m(f\star g) \mid H^1_{\mathrm{ad}}(K)\big), \qquad f^\sharp(x) = |x|^{-1}\overline{f(x^{-1})},$$

and $Q_K(f) = \langle f, f^\sharp\rangle_{\mathrm{ad}}$.

Theorem 11.19 (Connes–Consani–Marcolli [CCM09, Corollary 6.3, p. 29])**.** *Let K be a global field and let Q_K be as in Definition 11.18. Then $Q_K(f) \geq 0$ for every $f \in \mathbf{S}(C_K)$ if and only if every L-function with Grössencharakter on K satisfies the Riemann hypothesis.*

Proof. Put $f^*(x) = \overline{f(x^{-1})}$ and $\Delta^z f(x) = |x|^z f(x)$. The Weil positivity criterion, in the normalization of the source, is $\mathrm{Tr}(W(f\star f^*)) \geq 0$ for every test function, where $W(f) = \vartheta_m(\Delta^{-1/2} f)$ [CCM09, Proposition 6.2, p. 29]. For $h = \Delta^{-1/2} f$, one has $\Delta^{-1/2} f^* = \Delta^{-1} h^* = h^\sharp$; hence that trace is $Q_K(h)$. Since $\Delta^{-1/2}$ is an automorphism of the test algebra, the two positivity statements are equivalent. □

Theorem 11.20 (Adelic trace positivity implies RH)**.** *Let $Q_{\mathbb{Q}}$ be the trace form on $\mathbf{S}(C_{\mathbb{Q}})$ in Definition 11.18. If $Q_{\mathbb{Q}}(f) \geq 0$ for every $f \in \mathbf{S}(C_{\mathbb{Q}})$, then every nontrivial zero of $\zeta(s)$ has real part $1/2$.*

Proof. Apply Theorem 11.19 with $K = \mathbb{Q}$. Its hypothesis is the stated nonnegativity for the full test algebra, so every Grössencharakter L-function of $\mathbb{Q}$ satisfies RH. The trivial Grössencharakter has $L(s, 1) = \zeta(s)$, giving the claim. □

The hypothesis concerns every test function and every Hecke-character sector. Classical RH for ζ alone is not asserted to imply it; the full equivalence is Theorem 11.19. This is a criterion, not an established positive form: the source proves that the trace pairing vanishes on the range of its restriction map [CCM09, Proposition 6.4, pp. 29–30], but does not prove $Q_K \geq 0$ on all test functions. Its form is semidefinite if the criterion holds and has a radical; identifying it with the positive polarization (1.1) would require an additional comparison of spaces and scaling operators. The source also leaves the analogue of principal divisors open [CCM09, p. 4].

Remark 11.21 (Adelic realization and the full axioms). Let K be a global field. The C_K-action on $H^1_{\mathrm{ad}}(K)$ realizes the zero spectra of all Grössencharakter L-functions, and its integrated action has the trace formula of [CCM09, Proposition 4.13 and Theorem 4.16, pp. 23–24]. We call this an *adelic spectral realization*. For a number field K, its Hecke-character decomposition [CCM09, Proposition 4.12, p. 23] contains more sectors than the single function $Z_{\overline{X}_K} = \Lambda_K$ of

Definition 1.19. Even after restricting to the trivial-character sector, the cited construction does not give the cup product, endpoints, duality, and positive star required by Definition 1.5. In particular, Theorem 11.19 makes positivity of its trace form an RH-equivalent condition, not an established property. Thus the adelic realization is a concrete spectral candidate within the broader program, without being an instance of the full definition.

Remark 11.22 (The sense in which CCM has cohomology). Let K be a global field. CCM define a cokernel in cyclic modules and then apply cyclic homology to obtain $H^1(\mathbb{A}_K/K^\times, C_K)$. They explicitly interpret it as a first relative cohomology group for the inclusion of C_K in the adelic class space [CCM09, Definition 4.10 and (4.28), pp. 21–22]. We call this the *CCM adelic cyclic homology construction*. The cited paper does not construct the graded functor with products, endpoints, and duality of Definition 1.21, or a polarization; hence this relative H^1 alone is not a Deninger cohomology theory in the sense of these notes.

Theorem 11.23 ([CC17, Theorems 3.9, p. 14, and 5.17, p. 32])**.** *The points of the scaling topos* $[0,\infty) \rtimes \mathbb{N}^\times$ *are canonically identified with* $\mathcal{A}(\mathbb{R}_+^{\max})$*. For a prime* p *and a divisor* D *on the periodic orbit* C_p*, with the continuous dimension* $\dim_{\mathbb{R}}$ *of the source,* $\dim_{\mathbb{R}} H^0(D) - \dim_{\mathbb{R}} H^0(-D) = \deg D$*.*

This Riemann–Roch equality concerns divisors on a periodic orbit; it is not an intersection-positivity theorem on an arithmetic surface.

Theorem 11.24 ([CCM24, Proposition 4.8, p. 24])**.** *Let* $S \ni \infty$ *be a finite set of places of* $\mathbb{Q}$ *and* $\lambda > 0$*. The map* υ_S *of the source is an isomorphism of hilbertian spaces, in the source's terminology (a linear homeomorphism of topological vector spaces whose topology, but not a preferred inner product, comes from a Hilbert-space structure), from the semilocal Sonin space* $\mathcal{S}_\lambda(X_S, \alpha)$ *onto the space* $\mathcal{B}_\lambda$*, compatible with the semilocal amplification map.*

Giving $\mathcal{B}_\lambda$ the S-dependent induced norm defines the de Branges space $\mathcal{B}_\lambda^S$ of [CCM24, §4.8.3]. An isomorphism of abstract hilbertian spaces does not identify norms, embeddings, or operators on entire functions (Proposition 9.10).

11.5. **De Branges side.** Let $E = E_F = F + iF'$ (Definition 3.2). Since $F'(z) = -i\xi'(\frac{1}{2} - iz)$, one has $E(z) = \xi(\frac{1}{2} - iz) + \xi'(\frac{1}{2} - iz)$, which is the function denoted E_ξ in [Suz25, (1.3), p. 2]. Let $\vartheta = E^\#/E$ (denoted Θ_ξ there), let $\mathcal{K}(\vartheta) = H^2(\mathbb{C}_+) \ominus \vartheta H^2(\mathbb{C}_+)$ be the model space, and let $\mathcal{H}(E)$ be the de Branges space of Definition 9.29. When Q_W (Definition 3.9) is positive definite, let $\mathcal{H}_W$ be the Hilbert-space completion of $C_c^\infty(\mathbb{R})$ for the norm $\|\psi\|_W = Q_W(\psi)^{1/2}$, also written $\|\cdot\|_W$ on $\mathcal{H}_W$, and $[\psi] \in \mathcal{H}_W$ the class of ψ. Let F be the Fourier transform of Definition 3.9, extended to $L^2(\mathbb{R})$, and for $G \in L^2(\mathbb{R})$ put $\psi_G = \mathsf{F}^{-1}G$.

Theorem 11.25 ([Suz25, Theorem 1.1, p. 2])**.** *Assume RH. Then* $\mathcal{H}_W$ *is defined, and the map* $G \mapsto [\psi_G]$ *of the source is an isomorphism of Hilbert spaces* $\mathcal{K}(\vartheta) \to \mathcal{H}_W$*, with* $\|EG\|^2_{\mathcal{H}(E)} = \|G\|^2_{\mathcal{K}(\vartheta)} = \pi\|[\psi_G]\|^2_W$ *for* $G \in \mathcal{K}(\vartheta)$*.*

In the source, every class in $\mathcal{H}_W$ has a representative in $L^2(\mathbb{R})$ [Suz25, Theorem 5.5], $[\psi]$ denotes the class represented by $\psi \in L^2(\mathbb{R})$, and the source's Weil form $\langle\psi,\psi\rangle_W$ coincides with $Q_W(\psi)$ for $\psi \in C_c^\infty(\mathbb{R})$.

The self-adjoint operator of [Suz25, §6, p. 18] has each distinct zero as an eigenvalue of multiplicity one, whatever its order as a zero. The source's RH-equivalent norm criterion [Suz25, Theorem 1.4, p. 4] is not imported: with the printed definitions its norm expression degenerates (Proposition 9.34).

For $\omega > 0$ and $s = \frac{1}{2} - iz$, the source [Suz12, (1.1), p. 1] puts $A^\omega(z) = \frac{1}{2}(\xi(s+\omega) + \xi(s-\omega))$ and $B^\omega(z) = \frac{i}{2}(\xi(s+\omega) - \xi(s-\omega))$, so that $\xi(\frac{1}{2} + \omega - iz) = A^\omega(z) - iB^\omega(z)$; these are real

entire functions, and they are the real and imaginary parts of $\xi(\frac12+\omega-iz)$, up to sign, only for real z. For $\omega>1$ and $a>0$, let $\mathsf{H}_{\omega,a}$ be the operators of [Suz12, (2.6), p. 8]; when $1\pm\mathsf{H}_{\omega,a}$ are invertible, put $m_\omega(a)=\det(1+\mathsf{H}_{\omega,a})/\det(1-\mathsf{H}_{\omega,a})$ (Fredholm determinants).

Theorem 11.26 ([Suz12, Theorem 2.3, p. 8]). *Let $\omega>1$. Then the operators $\mathsf{H}_{\omega,a}$ are self-adjoint of Hilbert–Schmidt type, $1\pm\mathsf{H}_{\omega,a}$ are invertible for all $a>0$, and the canonical system with Hamiltonian $\operatorname{diag}(m_\omega(a)^{-2},m_\omega(a)^{2})$ of the source has an explicit solution (A_a,B_a) with $(A_1,B_1)=(A^\omega,B^\omega)$.*

Theorem 11.27 ([Suz21, Theorem 2.4, p. 9]). *For $\omega>0$ and $\nu\in\mathbb{Z}_{>0}$ with $\nu\omega>1$ let $K^{\omega,\nu}[t]$ and $J^{\omega,\nu}(t;z,z)$ be the Hankel-type operators and endpoint kernels of the source for $L=\zeta$. RH holds if and only if there is a sequence (ω_n,ν_n) with ω_n strictly decreasing to 0, $\nu_n\omega_n>1$, $\det(1\pm K^{\omega_n,\nu_n}[t])\neq0$ for all $t\geq0$, and $\lim_{t\to\infty}J^{\omega_n,\nu_n}(t;z,z)=0$ for all $z\in\mathbb{C}_+$.*

For $f\in C_c^\infty(\mathbb{R}_+^*)$ write $\widetilde{f}(s)=\int_0^\infty f(x)x^{s-1}\,dx$, $f^\sharp(x)=x^{-1}f(x^{-1})$, and $\bar f$ for the complex conjugate; the convolution is $(f*g)(u)=\int_0^\infty f(v)g(u/v)\,dv/v$. For a prime p put $W_p(f)=(\log p)\sum_{m\geq1}\big(f(p^m)+f^\sharp(p^m)\big)$, and put $W_{\mathbb{R}}(f)=(\log4\pi+\gamma_E)f(1)+\int_1^\infty\big(f(x)+f^\sharp(x)-\frac{2}{x}f(1)\big)\frac{dx}{x-x^{-1}}$, with γ_E Euler's constant.

Theorem 11.28 (Weil [Wei52], [Wei72, §16, p. 16]; [CC20, (148), p. 50, and Proposition C.1, p. 51]). *For $f\in C_c^\infty(\mathbb{R}_+^*)$ one has the explicit formula $\sum_\rho\widetilde{f}(\rho)=\int_0^\infty f(x)\,dx+\int_0^\infty f^\sharp(x)\,dx-\sum_v W_v(f)$, the sum on the left running over the nontrivial zeros of ζ, counted with multiplicity, and v over the places of $\mathbb{Q}$. Moreover, if $\mathcal{P}\subset\mathbb{C}$ is a finite set containing $\{0,1\}$ and disjoint from the nontrivial zeros, then RH holds if and only if $\sum_v W_v(g*\bar g^\sharp)\leq0$ for every $g\in C_c^\infty(\mathbb{R}_+^*)$ with $\widetilde{g}(z)=0$ for all $z\in\mathcal{P}$.*

The source states the explicit formula (148) without proof, following Bombieri, and its proof of Proposition C.1 refers to a proposition of Yoshida [Yos92, Proposition 1 and Lemma 1, p. 285], with a sketched refinement; the criterion is also recorded in [Bom06, §V, pp. 8–9], and in a strengthened form in [Bom00, Theorem 1, p. 191]. It is due to Weil [Wei52], as reported in [Bom06, §V, p. 9]; in [Wei72, §16, p. 16] Weil extends it to Artin–Hecke L-functions, where positivity is equivalent to the Riemann hypothesis together with Artin's conjecture. Proposition 12.7 translates it into the conventions of Definition 3.9.

Let $\xi_{\mathrm{CL}}(s)=s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)=2\xi(s)$, the normalization of [CL00].

Theorem 11.29 ([CL00, §3.1, (3.2) p. 6 and (3.4) p. 7]). *For the 34th nontrivial zero $\rho=\frac12+i\,111.0295355\ldots$ of ζ, $-\operatorname{Re}\{\xi'_{\mathrm{CL}}(\rho)\xi_{\mathrm{CL}}(1+\rho)\}=-5.389100507\ldots\times10^{-69}<0$, and $\operatorname{Re}\{\xi_{\mathrm{CL}}(1+282i)/\xi_{\mathrm{CL}}(2+282i)\}=-0.000131957\ldots<0$.*

These are high-precision numerical evaluations, not interval-certified bounds. In the normalization $\xi=\frac12\xi_{\mathrm{CL}}$ the first quantity is one quarter of the displayed value, about $-1.3472751268\times10^{-69}$; the second is a ratio and is independent of the normalization.

12. De Branges spaces and the Weil form

This section records what the analytic side can and cannot contribute to the positivity required in Theorem 6.3. Throughout, F is the function of Definition 1.16(d), and we use the following facts from Theorem 3.4: F is a real entire function ($F(\bar z)=\overline{F(z)}$), even, of order one; every zero of F lies in the open strip $\mathbb{S}=\{z\in\mathbb{C}:|\operatorname{Im}z|<\frac12\}$; and the number $N(R)$ of zeros of F in $|z|\leq R$, counted with multiplicity, satisfies $N(R)=O(R\log R)$ as $R\to\infty$, so that (3.2) holds. By Proposition 3.1, $F(0)\neq0$. Moreover, F has at least one zero: an even entire function of order one without zeros is e^{A+Bz} with $B=0$, hence constant, whereas $\xi(s)\to\infty$ as $s\to+\infty$ along the reals.

The Weil form Q_W and its spectral expression are introduced in Section 3.2 (Definition 3.9 and Proposition 3.10). By Proposition 3.10(iii), RH implies $Q_W \geq 0$. The converse, that nonnegativity of Q_W on a suitable class of tests forces all zeros to be real, is Weil's positivity criterion (Theorem 11.28, in the present conventions Proposition 12.7); it is used in Proposition 12.9.

12.1. **The Hermite–Biehler criterion.** Since F is even, $F(0) \neq 0$, and a zero r has the same order as $-r \neq r$, the zeros listed with multiplicity split into pairs $\{r, -r\}$. Fix a multiset R_+ containing one member of each pair.

Proposition 12.1. *Let F be as in Definition 1.16, let $\mathfrak{m} = -F'/F$, and let $E = E_F = F + iF'$ (Definition 3.2), so that $E^{\#} = F - iF'$. The following are equivalent.*

(i) Every zero of F is real (the Riemann hypothesis).
(ii) $\mathfrak{m}$ is holomorphic on $\mathbb{C}_+$ and $\operatorname{Im}\mathfrak{m}(z) > 0$ for every $z \in \mathbb{C}_+$.
(iii) $\operatorname{Im}\mathfrak{m}(z) \geq 0$ for every $z \in \mathbb{C}_+$ with $F(z) \neq 0$.
(iv) E satisfies the HB inequality of Definition 3.2.

For this E the equivalence of the Riemann hypothesis with the HB inequality is Lagarias's theorem [Lag06a, Theorem 1, p. 9], whose proof passes through the Herglotz condition [Lag06a, (10), p. 9], which Lagarias takes from [Lag99]; Proposition 12.1 is reproduced to keep multiplicities explicit. Common real zeros of E and $E^{\#}$ are allowed in (iv). The proof uses two identities. First, the product and partial-fraction expansions

$$F(z) = F(0)\prod_{r\in R_+}\Big(1-\frac{z^2}{r^2}\Big), \qquad \mathfrak{m}(z) = \sum_{r\in R_+}\Big(\frac{1}{r-z}+\frac{1}{-r-z}\Big), \tag{12.1}$$

the product converging absolutely and locally uniformly on $\mathbb{C}$ and the series absolutely and locally uniformly on $\mathbb{C}$ minus the zeros of F. Second, the identity of entire functions

$$|F+iF'|^2 - |F-iF'|^2 = 4\operatorname{Im}\big(F\,\overline{F'}\big) = 4|F|^2\operatorname{Im}\Big(-\frac{F'}{F}\Big) \quad \text{where } F \neq 0. \tag{12.2}$$

Proof of (12.1) and (12.2). Since $F(\bar z) = \overline{F(z)}$, one has $F^{\#} = F$ and $(F')^{\#} = F'$, so $E^{\#} = F - iF'$. By Hadamard's factorization theorem for the order-one function F, $F(z) = e^{A+Bz}\prod_n(1 - z/r_n)e^{z/r_n}$, and by (3.2) this product converges absolutely and locally uniformly. Absolute convergence allows grouping r with $-r$, and $(1 - z/r)e^{z/r}(1 + z/r)e^{-z/r} = 1 - z^2/r^2$; thus $F = e^{A+Bz}G$ with G the even product in (12.1). Evenness of F and G gives $e^{2Bz} = 1$ off the zeros of G, so $B = 0$, and $e^A = F(0)$. Taking the logarithmic derivative of the locally uniformly convergent product termwise and using $\frac{d}{dz}\log(1 - z^2/r^2) = -2z/(r^2 - z^2)$ and $\frac{1}{r-z} - \frac{1}{r+z} = \frac{2z}{r^2-z^2}$ gives the series for $\mathfrak{m}$; for $|r| \geq 2|z|$ its terms are bounded by $8|z|/(3|r|^2)$, so (3.2) gives locally uniform absolute convergence. For (12.2), $|a + b|^2 - |a - b|^2 = 4\operatorname{Re}(a\bar b)$ with $a = F$, $b = iF'$ gives $4\operatorname{Re}(-iF\overline{F'}) = 4\operatorname{Im}(F\overline{F'})$, and where $F \neq 0$ one has $\operatorname{Im}(F\overline{F'}) = |F|^2\operatorname{Im}(\overline{F'}/\overline{F}) = |F|^2\operatorname{Im}(-F'/F)$. □

Proof of (i)⇒(ii). If all zeros are real then F has no zero in $\mathbb{C}_+$ and $\mathfrak{m}$ is holomorphic there. For real r and $z = x + iy$ with $y > 0$, $\operatorname{Im}\frac{1}{r-z} = \frac{y}{(x-r)^2+y^2} > 0$. Taking imaginary parts in (12.1),

$$\operatorname{Im}\mathfrak{m}(z) = \sum_{\gamma\in\Gamma} m_\gamma \frac{y}{(x-\gamma)^2+y^2},$$

a convergent series of positive terms; it is nonempty because F has a zero. □

Proof of (ii)⇒(iii). Immediate. □

Proof of (iii)⇒(i). Suppose $F(z_0)=0$ with $z_0\in\mathbb{C}_+$, of order $k\ge 1$. Write $F=(z-z_0)^k g$ near z_0 with $g(z_0)\neq 0$, so that $\mathfrak{m}(z)=-k/(z-z_0)-g'(z)/g(z)$ and $|g'/g|\le M$ on a disc $|z-z_0|\le\eta$ with $0<\eta<\operatorname{Im} z_0$ on which z_0 is the only zero of F. For $z=z_0-i\epsilon$, $0<\epsilon<\eta$, we have $z\in\mathbb{C}_+$, $F(z)\neq 0$, and $-k/(z-z_0)=-ik/\epsilon$, so $\operatorname{Im}\mathfrak{m}(z)\le -k/\epsilon+M<0$ once $\epsilon<k/M$. This contradicts (iii); hence F has no zero in $\mathbb{C}_+$. If $F(z_0)=0$ with $\operatorname{Im} z_0<0$ then $F(\bar z_0)=\overline{F(z_0)}=0$ with $\bar z_0\in\mathbb{C}_+$, which is excluded. So every zero is real. □

Proof of (ii)⇒(iv). At a zero of F of order k, $\mathfrak{m}$ has a simple pole with residue $-k$; so holomorphy of $\mathfrak{m}$ on $\mathbb{C}_+$ means $F\neq 0$ on $\mathbb{C}_+$. By (12.2), $|E|^2-|E^{\#}|^2=4|F|^2\operatorname{Im}\mathfrak{m}>0$ on $\mathbb{C}_+$. □

Proof of (iv)⇒(i). If $F(z_0)=0$ with $z_0\in\mathbb{C}_+$, then $E(z_0)=iF'(z_0)$ and $E^{\#}(z_0)=-iF'(z_0)$ have equal moduli (both zero if the zero is multiple), contradicting the strict HB inequality at z_0. Hence F has no zero in $\mathbb{C}_+$, and by reality none in the lower half-plane. □

Remark 12.2. Let r be a real zero of F of order k, and write $F=(z-r)^k g$ with $g(r)\neq 0$. Then $E=(z-r)^{k-1}\big[(z-r)g+i(kg+(z-r)g')\big]$, and the bracket equals $ikg(r)\neq 0$ at $z=r$; likewise for $E^{\#}$. So a real zero of order k produces a common real zero of E and $E^{\#}$ of order exactly $k-1$. Proposition 12.1 is therefore an equivalence with RH itself, not with RH together with simplicity of the zeros; a convention forbidding real zeros of the structure function would, for this E, add simplicity [Lag06a, §1]. Discarding the common factor loses divisor data (Proposition 9.8). Under RH the expression for $\operatorname{Im}\mathfrak{m}$ in the proof is the Poisson integral of the measure $\sum_\gamma m_\gamma\delta_\gamma$: multiplicities enter as weights, not as dimensions of eigenspaces. In the variable $s=\frac12-iz$ one has $F'(z)=-i\xi'(s)$, so $\mathfrak{m}(z)=i\xi'(s)/\xi(s)$ and $\operatorname{Im}\mathfrak{m}(z)=\operatorname{Re}(\xi'(s)/\xi(s))$; since $\operatorname{Im} z>0$ exactly when $\operatorname{Re} s>\frac12$, condition (ii) reads $\operatorname{Re}(\xi'/\xi)>0$ on $\operatorname{Re} s>\frac12$ [Lag06a, (10), p. 9].

Proposition 12.3. *Let F be as in Definition 1.16 and $\mathfrak{m}=-F'/F$. Then F has no zero in the closed half-plane $\{\operatorname{Im} z\ge\frac12\}$, and $\operatorname{Im}\mathfrak{m}(z)>0$ for every z with $\operatorname{Im} z\ge\frac12$.*

Proof. All zeros lie in $\mathbb{S}$, so $F\neq 0$ on $\operatorname{Im} z\ge\frac12$. For a zero r and $z=x+iy$, $\operatorname{Im}\frac{1}{r-z}=\frac{y-\operatorname{Im} r}{|r-z|^2}$, and likewise $\operatorname{Im}\frac{1}{-r-z}=\frac{y+\operatorname{Im} r}{|r+z|^2}$. If $y\ge\frac12$ then $y\mp\operatorname{Im} r>0$ because $|\operatorname{Im} r|<\frac12$. So every term of the series (12.1) has positive imaginary part, and so does the convergent sum. □

Proposition 12.3 is unconditional, but its argument uses the location of the zeros term by term and gives nothing on $0<\operatorname{Im} z<\frac12$. A zero z_0 with $0<\operatorname{Im} z_0<\frac12$ makes $\operatorname{Im}\mathfrak{m}$ negative just below z_0 (proof of (iii)⇒(i)), while $\operatorname{Im}\mathfrak{m}$ remains positive above $\frac12$; an explicit even real polynomial exhibiting both behaviours is given in Proposition 9.12. Positivity high in $\mathbb{C}_+$ therefore does not propagate down to the real axis.

12.2. **Canonical systems.** Let $J_0=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ and let $H:[0,\infty)\to M_2(\mathbb{R})$ be locally integrable with $H(t)$ symmetric and positive semidefinite for almost every t. For $z\in\mathbb{C}$ consider

$$\partial_t Y(t,z)=zJ_0H(t)Y(t,z),\qquad Y(0,z)=(1,0)^T. \tag{12.3}$$

The Picard series $Y=\sum_{n\ge0}z^nY_n$, with $Y_0=(1,0)^T$ and $Y_n(t)=\int_0^t J_0H(u)Y_{n-1}(u)\,du$, satisfies $|Y_n(t)|\le(\int_0^t\|H\|)^n/n!$; hence it defines a locally absolutely continuous solution in t which is entire in z, and Gronwall's inequality shows it is the unique locally absolutely continuous solution, also backwards from any initial time. Write $Y=(A,B)^T$ and $E_t(z)=A(t,z)-iB(t,z)$.

Proposition 12.4. *Let H, J_0, and Y be as in (12.3), and let $t\ge0$ and $z,w\in\mathbb{C}$. Then:*

(i) $Y(t,\bar z)=\overline{Y(t,z)}$; hence $E_t^{\#}=A+iB$;

(ii) for almost every t,

$$\partial_t\big(Y(t,w)^*J_0Y(t,z)\big)=(\bar w-z)\,Y(t,w)^*H(t)Y(t,z); \tag{12.4}$$

(iii)

$$|E_t(z)|^2 - |E_t^{\#}(z)|^2 = 4\,\mathrm{Im}\,z \int_0^t Y(u,z)^* H(u) Y(u,z)\,du; \tag{12.5}$$

(iv) if $H(u)$ is positive definite for almost every u and $t > 0$, then E_t satisfies the HB inequality.

Proof of (i). Since H and J_0 are real, $\overline{Y(\cdot,z)}$ solves (12.3) with $\bar z$ in place of z and the same initial value; uniqueness gives the first claim. Then $E_t^{\#}(z) = \overline{A(t,\bar z)} + i\,\overline{B(t,\bar z)} = A(t,z) + iB(t,z)$. □

Proof of (ii). Since $J_0^* = J_0^T = -J_0$, $J_0^2 = -I$, and $H^* = H$, the adjoint of $\partial_t Y(t,w) = wJ_0HY(t,w)$ is $\partial_t Y(t,w)^* = -\bar w Y(t,w)^* H J_0$. Products of locally absolutely continuous functions are locally absolutely continuous, and almost everywhere

$$\partial_t(Y_w^* J_0 Y_z) = -\bar w Y_w^* H J_0^2 Y_z + z Y_w^* J_0^2 H Y_z = (\bar w - z) Y_w^* H Y_z,$$

where $Y_w = Y(t,w)$ and $Y_z = Y(t,z)$. □

Proof of (iii). For $Y = (A,B)^T$ one computes $Y^* J_0 Y = -\bar A B + \bar B A = 2i\,\mathrm{Im}(A\bar B)$, which vanishes at $t = 0$. Integrating (12.4) with $w = z$, $2i\,\mathrm{Im}(A\bar B)(t,z) = (\bar z - z)\int_0^t Y^* H Y\,du$, that is, $\mathrm{Im}(A\bar B) = -\,\mathrm{Im}\,z \int_0^t Y^* H Y\,du$. By (i), $|E_t|^2 - |E_t^{\#}|^2 = |A - iB|^2 - |A + iB|^2 = 4\,\mathrm{Re}(A\,\overline{-iB}) = -4\,\mathrm{Im}(A\bar B)$. □

Proof of (iv). If $Y(u_0,z) = 0$ for some u_0, uniqueness from u_0 gives $Y(\cdot,z) \equiv 0$, contradicting $Y(0,z) = (1,0)^T$. Hence $Y(u,z)^* H(u) Y(u,z) > 0$ for almost every u, the integral in (12.5) is positive for $t > 0$, and (12.5) gives $|E_t(z)| > |E_t^{\#}(z)|$ for $z \in \mathbb{C}_+$. □

For $H = I$ one has $Y(t,z) = (\cos zt, \sin zt)^T$ and $E_t(z) = e^{-izt}$; with $y = \mathrm{Im}\,z$, both sides of (12.5) equal $2\sinh(2ty)$, since $|\cos w|^2 + |\sin w|^2 = \cosh(2\,\mathrm{Im}\,w)$.

Proposition 12.4 shows that a positive Hamiltonian on an initial interval produces HB functions E_t. To reach $\Xi + i\Xi'$, or a shifted ξ, one must in addition prove the endpoint limit, nonvanishing, and normalization, and the relevant domain properties. Conversely, an inverse theorem for canonical systems whose input is an HB function cannot be applied to $\Xi + i\Xi'$ without first proving Proposition 12.1(iv) (Proposition 9.12).

It is natural to expect that a canonical system with positive Hamiltonian on an initial interval, whose solutions are entire, already forces the HB inequality for the function it is meant to produce at the endpoint; in fact positivity on an initial interval imposes no sign on a prescribed terminal function.

Remark 12.5. Let H be as in (12.3) on $[0,t_0]$ and let $t_1 > t_0$. Integrating (12.4) from t_0 to t_1 gives

$$|E_{t_1}(z)|^2 - |E_{t_1}^{\#}(z)|^2 = |E_{t_0}(z)|^2 - |E_{t_0}^{\#}(z)|^2 + 4\,\mathrm{Im}\,z \int_{t_0}^{t_1} Y^* H Y\,du,$$

so the sign at t_1 is controlled only if $H \geq 0$ on all of $[t_0,t_1]$; at an infinite endpoint, the terminal boundary term $\lim_{t\to\infty}(|E_t|^2 - |E_t^{\#}|^2)$ must itself be identified and controlled. Positivity of H on $[0,t_0]$ says nothing about the Hamiltonian, if any, that continues the system to an endpoint with a prescribed terminal function such as $\Xi + i\Xi'$.

Suzuki constructs, for the shifted functions $\xi(\frac12 + \omega - iz)$ with $\omega > 1$, canonical systems with positive diagonal Hamiltonians whose solutions match the shifted function (Theorem 11.26); his systems are written in a multiplicative variable a with operator $-a\partial_a$, so comparison with (12.3) requires a change of variable and a reversal of time. For powers $\xi(\frac12 + \omega - iz)^\nu$ with $\nu\omega > 1$ he obtains a positive Hamiltonian on some initial interval unconditionally, and shows that RH is

equivalent to the existence of shifts $\omega_n \downarrow 0$ and integers ν_n with $\nu_n\omega_n > 1$ for which the relevant determinants do not vanish for all $t \geq 0$ and an endpoint kernel tends to zero (Theorem 11.27).

It is natural to expect that these results, pushed to small shifts or large exponents, reach ξ itself; in fact the checked range stops short of it.

Remark 12.6. Theorem 11.26 covers only shifts $\omega > 1$, where the shifted function has no zeros in $\mathbb{C}_+$ unconditionally; it does not give the corresponding system for every $\omega > 0$. In Theorem 11.27, increasing ν improves the regularity of the kernel but does not prove RH: the content of the criterion is the global invertibility $\det(1 \pm K^{\omega_n,\nu_n}[t]) \neq 0$, for all $t \geq 0$, of the kernel operators of that theorem, together with the endpoint condition, and the unconditional part is the existence of the system on an initial interval.

12.3. **Norm identities and the converse.** Let $\mathcal{T}_W = \{\psi \in C_c^\infty(\mathbb{R}) : \widehat{\psi}(\frac{i}{2}) = \widehat{\psi}(-\frac{i}{2}) = 0\}$.

Proposition 12.7 (Weil)**.** *Let F and Q_W be as in Definition 3.9. Then every zero of F is real if and only if $Q_W(\psi) \geq 0$ for every $\psi \in \mathcal{T}_W$.*

Proof. For $\psi \in C_c^\infty(\mathbb{R})$ put $g(x) = x^{-1/2}\psi(\log x)$; this is a bijection $C_c^\infty(\mathbb{R}) \to C_c^\infty(\mathbb{R}_+^*)$. With the notation of Theorem 11.28 and $x = e^t$, $\widetilde{g}(\frac{1}{2} - iz) = \int_{\mathbb{R}} \psi(t)e^{-izt}\,dt = \widehat{\psi}(-z)$. Hence $\widetilde{g}(0) = \widehat{\psi}(\frac{i}{2})$ and $\widetilde{g}(1) = \widehat{\psi}(-\frac{i}{2})$, so g satisfies the vanishing conditions for $\mathcal{P} = \{0, 1\}$ exactly when $\psi \in \mathcal{T}_W$; and $\{0, 1\}$ contains no nontrivial zero. Let $f = g * \bar{g}^\sharp$. The Mellin transform turns $*$ into the product, and the substitution $x = 1/y$ gives $\widetilde{\bar{g}^\sharp}(s) = \overline{\widetilde{g}(1 - \bar{s})}$; thus $\widetilde{f}(s) = \widetilde{g}(s)\overline{\widetilde{g}(1 - \bar{s})}$. In particular $\int_0^\infty f\,dx = \widetilde{f}(1) = 0$ and $\int_0^\infty f^\sharp\,dx = \widetilde{f}(0) = 0$ when $\psi \in \mathcal{T}_W$. A nontrivial zero ρ corresponds to the zero z of F with $\rho = \frac{1}{2} - iz$, with the same multiplicity, and then $1 - \bar{\rho} = \frac{1}{2} - i\bar{z}$. The explicit formula of Theorem 11.28 therefore gives

$$-\sum_v W_v(g * \bar{g}^\sharp) = \sum_z \widehat{\psi}(-z)\overline{\widehat{\psi}(-\bar{z})} = \sum_z \widehat{\psi}(z)\overline{\widehat{\psi}(\bar{z})} = Q_W(\psi),$$

the sums running over the zeros of F with multiplicity; the middle equality uses that the zeros of the even function F are symmetric under $z \mapsto -z$ with multiplicity, and the rearrangement is justified by absolute convergence (Proposition 3.10). The criterion of Theorem 11.28 is thus exactly the statement. □

Definition 12.8. Let Q_W be as in Definition 3.9 and let $\mathcal{T}_0 \subset C_c^\infty(\mathbb{R})$ be a linear subspace. A *norm realization* of Q_W on $\mathcal{T}_0$ is a pair $(\mathcal{K}, \mathcal{T})$ of a Hilbert space $\mathcal{K}$ and a linear map $\mathcal{T} : \mathcal{T}_0 \to \mathcal{K}$ with $\|\mathcal{T}\psi\|^2 = \pi Q_W(\psi)$ for every $\psi \in \mathcal{T}_0$.

Proposition 12.9. *Let F and Q_W be as in Definition 3.9, let $\mathcal{T}_W$ be as in Proposition 12.7, and suppose that Q_W has a norm realization $(\mathcal{K}, \mathcal{T})$ on a subspace $\mathcal{T}_0 \supset \mathcal{T}_W$. Then every zero of F is real.*

Proof. For $\psi \in \mathcal{T}_W \subset \mathcal{T}_0$, $Q_W(\psi) = \pi^{-1}\|\mathcal{T}\psi\|^2 \geq 0$. Proposition 12.7 applies. □

Linearity of $\mathcal{T}$ is not used in the proof. A norm realization on all of $C_c^\infty(\mathbb{R})$ restricts to $\mathcal{T}_W$; one defined only on a smaller class, for instance on even tests, suffices only if that class contains $\mathcal{T}_W$. The hypothesis cannot be weakened to a norm identity on a single support window (Proposition 9.16). Under RH a norm realization on $C_c^\infty(\mathbb{R})$ exists trivially: $\psi \mapsto (\sqrt{\pi m_\gamma}\,\widehat{\psi}(\gamma))_\gamma$ lands in $\ell^2(\Gamma)$ by Proposition 3.10 and has the required norm. As with Proposition 6.6, the content lies in constructing $\mathcal{T}$ and $\mathcal{K}$ independently of the location of the zeros.

12.4. **Suzuki's comparison.** Let $E = E_F = F + iF'$, $\vartheta = E^{\#}/E$, $\mathcal{K}(\vartheta)$, $\mathcal{H}_W$, $\|\cdot\|_W$, and $\psi_G = \mathsf{F}^{-1}G$ be as in Section 11, before Theorem 11.25. Under RH, E satisfies the HB inequality (Proposition 12.1), so E has no zero in $\mathbb{C}_+$ and $|\vartheta| < 1$ there. Theorem 11.25 [Suz25, Theorem 1.1] states that, assuming RH, the Hilbert space $\mathcal{H}_W$ obtained by completing $C_c^\infty(\mathbb{R})$ for Q_W is identified with $\mathcal{K}(\vartheta)$ through the inverse Fourier transform, with $\|G\|^2_{\mathcal{K}(\vartheta)} = \pi\|[\psi_G]\|^2_W$. This is an identification of Hilbert spaces under RH, not a construction of a positive space from which RH would follow.

The same source proposes an explicit transform realizing the norm identity for every test. With the printed definitions, which extend its time-dependent kernels evenly to negative t, the transform vanishes on every even test, so the norm it defines is degenerate (Proposition 9.34). The following reconstruction defines the zero series for all real t without even extension. Since $m_\gamma = m_{\bar\gamma}$, it agrees for $t < 0$ with the rule $P_t(z) = \overline{P_{-t}(\bar z)}$ of the author's correction [Suz26, p. 27], which also replaces the factor i in the source's (1.5) by $-i$ and so multiplies S_t by -1.

For $t \in \mathbb{R}$ and $z \in \mathbb{C}$ away from the zeros of F put

$$P_t^{\mathrm{rec}}(z) = \sum_{\gamma\in\Gamma} m_\gamma \frac{e^{-i\gamma t} - 1}{\gamma(z-\gamma)}, \qquad S_t^{\mathrm{rec}}(z) = \frac{i\big(1+\vartheta^{\#}(z)\big)}{2}\, P_t^{\mathrm{rec}}(z). \tag{12.6}$$

The series converges absolutely, uniformly for t in a compact set and z in a compact set disjoint from Γ. Indeed, since $|\operatorname{Im}\gamma| < \frac12$, one has $|e^{-i\gamma t} - 1| \le 2e^{|t|/2}$; for z in a compact set, $|z-\gamma| \ge |\gamma|/2$ for all but finitely many γ; so all but finitely many terms are bounded by $4e^{|t|/2} m_\gamma |\gamma|^{-2}$, and $\sum_\gamma m_\gamma|\gamma|^{-2} < \infty$ by (3.2). This is the justification given after [Suz25, (3.2), p. 7] for $t \ge 0$. For $\psi \in C_c^\infty(\mathbb{R})$ define

$$\mathcal{T}_{\mathrm{rec}}\psi = \int_{\mathbb{R}} (S_t^{\mathrm{rec}})^{\#}\, i\psi'(t)\, dt. \tag{12.7}$$

Since $\overline{\vartheta^{\#}(\bar z)} = \vartheta(z)$ and $\int_{\mathbb{R}} (e^{i\bar\gamma t} - 1)\, i\psi'(t)\, dt = \widehat{i\psi'}(\bar\gamma) = \bar\gamma\widehat{\psi}(\bar\gamma)$, exchanging sum and integral (justified by the same majorant, since ψ' has compact support) and reindexing by $\gamma \mapsto \bar\gamma$ gives, pointwise off Γ and wherever ϑ is defined,

$$\mathcal{T}_{\mathrm{rec}}\psi = -\sum_{\gamma\in\Gamma} \sqrt{\pi m_\gamma}\, \widehat{\psi}(\gamma) f_\gamma, \qquad f_\gamma(z) = \sqrt{\frac{m_\gamma}{\pi}}\, \frac{i\big(1+\vartheta(z)\big)}{2(z-\gamma)}. \tag{12.8}$$

The functions f_γ are those of [Suz25, (3.5), p. 10], and the overall sign of (12.8) is opposite to that of [Suz25, (3.8), p. 10], which does not affect norms. Under RH, the f_γ form an orthonormal basis of $\mathcal{K}(\vartheta)$ [Suz25, Proposition 4.1, p. 10], an imported statement used only in this paragraph; it makes the series (12.8) converge in $L^2(\mathbb{R})$ with $\|\mathcal{T}_{\mathrm{rec}}\psi\|_2^2 = \pi Q_W(\psi)$. Without RH, neither the L^2 boundary behaviour of $\mathcal{T}_{\mathrm{rec}}\psi$ nor a prime-side formula for negative t is available.

Question 12.10. Let $\mathcal{T}_{\mathrm{rec}}$ be defined by (12.6) and (12.7). Without assuming RH, is $\mathcal{T}_{\mathrm{rec}}\psi$ the boundary value of a function whose restriction to $\mathbb{R}$ lies in $L^2(\mathbb{R})$, for every $\psi \in C_c^\infty(\mathbb{R})$, and does $\|\mathcal{T}_{\mathrm{rec}}\psi\|^2_{L^2(\mathbb{R})} = \pi Q_W(\psi)$ hold for every such ψ?

An affirmative answer to Question 12.10 would imply RH by Proposition 12.9, provided the Hilbert space is $L^2(\mathbb{R})$ with its own (positive) norm. By the agreement just noted, Question 12.10 concerns the corrected definition of the source; the unconditional norm identity is also stated as a target in [Suz26, pp. 26–27].

Multiplicities. Under RH, Proposition 3.10 gives one evaluation coordinate $\widehat{\psi}(\gamma)$ per distinct zero, carrying weight m_γ. The self-adjoint operator specified in [Suz25, §6] has multiplicity one at each distinct zero. The ordinary trace of f of an operator on this scalar completion is $\sum_\gamma f(\gamma)$, whereas the divisor identity of AW2, the determinant identity (AW-det), and the

trace in AW6 require $\sum_\gamma m_\gamma f(\gamma)$. A commuting weight operator, a weighted trace, or a Hilbert space with fibres $\mathbb{C}^{m_\gamma}$ would supply the difference; these are distinct constructions, each needing independent arithmetic meaning. A cyclic self-adjoint operator cannot have a two-dimensional eigenspace (Proposition 9.8). For zero location alone, an independently established scalar norm identity suffices (Proposition 12.9).

Semilocal spaces. For a finite set S of places containing ∞ and a cutoff $\lambda > 0$, Connes, Consani, and Moscovici construct from Sonin spaces an isomorphism of hilbertian spaces (a linear homeomorphism; no preferred inner product is transported) onto a space $\mathcal{B}_\lambda$ of entire functions, which with its S-dependent norm is a de Branges space $\mathcal{B}_\lambda^S$ (Theorem 11.24). This comparison involves arithmetic places, entire functions, and positivity unconditionally. It does not identify $\mathcal{B}_\lambda^S$ with $\mathcal{K}(\vartheta)$ or $\mathcal{H}_W$, nor prove convergence as S and λ grow; an abstract isomorphism of Hilbert spaces is too weak to transport spectra (Proposition 9.10).

12.5. **Spectral approximation.** For $\ell > 0$ we regard $L^2([-\ell,\ell])$ as the subspace of $L^2(\mathbb{R})$ of functions vanishing outside $[-\ell,\ell]$; its elements are integrable with compact support, so their Fourier transforms (Definition 3.9) are entire.

Definition 12.11. Let $\ell > 0$.

(a) A *gap datum* on $[-\ell,\ell]$ is a tuple $(\mathcal{E}, A, v, \mu)$ with $\mathcal{E} \subset L^2([-\ell,\ell])$ a finite-dimensional subspace, A a self-adjoint operator on $\mathcal{E}$, $v \in \mathcal{E}$ a unit eigenvector of A, and $\mu \in \mathbb{R}$, such that $d = \operatorname{dist}\bigl(\mu, \operatorname{spec}(A|_{v^\perp})\bigr) > 0$, where $v^\perp$ is the orthogonal complement of v in $\mathcal{E}$ (and $d = +\infty$ if $v^\perp = 0$).

(b) For $k \in \mathcal{E}$, its *residual* is $r = \|(A-\mu)k\|_2$ and its *projection* is $w = Pk$, where P is the orthogonal projection onto $\mathbb{C}v$.

(c) For $b \geq 0$ put $C(\ell, b) = \sqrt{\sinh(2b\ell)/b}$ if $b > 0$ and $C(\ell, 0) = \sqrt{2\ell}$.

The subspace $v^\perp$ is A-invariant because v is an eigenvector and A is self-adjoint, so $\operatorname{spec}(A|_{v^\perp})$ is defined.

Proposition 12.12. *Let $\ell > 0$, let $(\mathcal{E}, A, v, \mu)$ be a gap datum on $[-\ell,\ell]$ with gap d, and let $k \in \mathcal{E}$ have residual r and projection w. Then*

$$\|k - w\|_2 \leq r/d, \tag{12.9}$$

and for every $b \geq 0$,

$$\sup_{|\operatorname{Im} z| \leq b} \bigl|\widehat{k}(z) - \widehat{w}(z)\bigr| \leq C(\ell, b)\, r/d. \tag{12.10}$$

Proof. Choose an orthonormal eigenbasis $v = e_0, e_1, \dots, e_n$ of A with eigenvalues $\lambda_0, \dots, \lambda_n$, and write $k = \sum_i c_i e_i$. Then $w = c_0 e_0$ and $r^2 = \sum_i |\lambda_i - \mu|^2 |c_i|^2 \geq d^2 \sum_{i \geq 1} |c_i|^2 = d^2 \|k - w\|_2^2$, whatever c_0 and λ_0 are; this is (12.9). For $g \in L^2([-\ell,\ell])$ and $z = x + iy$, the Cauchy–Schwarz inequality gives

$$|\widehat{g}(z)|^2 \leq \|g\|_2^2 \int_{-\ell}^{\ell} e^{-2yt}\, dt = \|g\|_2^2\, \frac{\sinh(2y\ell)}{y}$$

for $y \neq 0$, and $\|g\|_2^2 \cdot 2\ell$ for $y = 0$. The function $y \mapsto \sinh(2y\ell)/y$ is even, with a power series in y^2 having positive coefficients and value 2ℓ at $y = 0$, so on $|y| \leq b$ it is at most $C(\ell, b)^2$. Applying this to $g = k - w$ and using (12.9) proves (12.10). □

We use Hurwitz's theorem in the following form: if $f_j \to f$ uniformly on a closed disc D inside a domain where all f_j and f are holomorphic, and f has a zero in the interior of D and none on ∂D, then f_j has a zero in D for all large j. (By Rouché's theorem, once $\sup_{\partial D} |f_j - f| < \min_{\partial D} |f|$.)

Proposition 12.13. *Let F be as in Definition 1.16 and $\mathbb{S} = \{|\operatorname{Im} z| < \frac{1}{2}\}$. For each $j \geq 1$ let $\ell_j > 0$, let $(\mathcal{E}_j, A_j, v_j, \mu_j)$ be a gap datum on $[-\ell_j, \ell_j]$ with gap d_j, and let $k_j \in \mathcal{E}_j$ have residual r_j and projection w_j. Assume:*

(a) every zero of $\widehat{v}_j$ is real, for every j;
(b) $\widehat{k}_j \to F$ uniformly on compact subsets of $\mathbb{S}$;
(c) $C(\ell_j, b)\, r_j/d_j \to 0$ as $j \to \infty$, for every $0 < b < \frac{1}{2}$.

Then every zero of F is real.

Proof. Let $L \subset \mathbb{S}$ be compact; then $L \subset \{|\operatorname{Im} z| \leq b\}$ for some $0 < b < \frac{1}{2}$. By Proposition 12.12 and (c), $\widehat{k}_j - \widehat{w}_j \to 0$ uniformly on L, so by (b), $\widehat{w}_j \to F$ uniformly on L. Since $F(0) > 0$ (Proposition 3.1), $\widehat{w}_j(0) \neq 0$ for large j; as $w_j = \langle k_j, v_j\rangle v_j$, this forces $\langle k_j, v_j\rangle \neq 0$, and then $\widehat{w}_j$ and $\widehat{v}_j$ have the same zeros, all real by (a). Suppose $F(z_0) = 0$ with $z_0 \notin \mathbb{R}$. Then $z_0 \in \mathbb{S}$. Choose η with $0 < \eta < \min(|\operatorname{Im} z_0|, \frac{1}{2} - |\operatorname{Im} z_0|)$ such that F has no zero on $|z - z_0| = \eta$ (zeros of $F \not\equiv 0$ are isolated). The closed disc $D = \{|z - z_0| \leq \eta\}$ lies in $\mathbb{S}$ and does not meet $\mathbb{R}$. By Hurwitz's theorem, $\widehat{w}_j$ has a zero in D for all large j, a nonreal zero, contradicting what was shown. Hence every zero of F, all of which lie in $\mathbb{S}$, is real. □

The argument uses neither $\ell_j \to \infty$, which is the regime in which the hypotheses are expected to be attainable, nor any assumption on the multiplicities of the real zeros of F; and (b) may be weakened to $\widehat{k}_j \to cF$ with a constant $c \neq 0$, which absorbs normalization factors such as the factor 4 of Proposition 9.32. Since $C(\ell, b) \sim e^{b\ell}/\sqrt{2b}$ as $\ell \to \infty$, condition (c) asks, in terms of $\lambda_j = e^{\ell_j}$, that $\lambda_j^b r_j/d_j \to 0$ for every $b < \frac{1}{2}$; the estimate $r_j/d_j = O(\lambda_j^{-1/2})$ suffices. Neither a small residual without a gap, nor unweighted L^2 convergence on growing supports, nor approximation in norm without graph-norm control of an unbounded operator gives (b)–(c) (Proposition 9.20).

Hypothesis (a) is where the arithmetic enters. In the first arXiv version of [CCM25, Theorem 5.10], the compression of the Weil quadratic form to periodic Fourier modes on an interval is a Hermitian matrix, and, if its lowest eigenvalue is simple, a corresponding eigenvector v is even, and a boundary functional does not vanish on v, then $\widehat{v}$ has only real zeros. The positivity used there is that of $Q - \varepsilon_{\min} I$ on a quotient, which does not give positivity of the Weil form itself (Proposition 9.16). What remains open along this route is a bound on r_j/d_j satisfying (c), together with verification of the simplicity, evenness, and boundary hypotheses along an unbounded sequence of intervals.

12.6. **Two comparison targets.** The discussion above isolates two targets of different strength.

Question 12.14. Let Q_W be as in Definition 3.9 and $\mathcal{T}_W$ as in Proposition 12.7. Construct from arithmetic or adelic data, without using the location of the zeros of F, a norm realization $(\mathcal{K}, \mathcal{T})$ of Q_W (Definition 12.8) on a subspace containing $\mathcal{T}_W$, preferably on all of $C_c^\infty(\mathbb{R})$, in which the positivity of the norm of $\mathcal{K}$ is independent of RH.

By Proposition 12.9 an answer to Question 12.14 implies RH; polarization then determines the sesquilinear Weil pairing. The transform $\mathcal{T}_{\mathrm{rec}}$ of Question 12.10 is one candidate; the literal transform of Proposition 9.34 is not. This target concerns zero location only and need not resolve multiplicities.

Question 12.15. Let $\overline{X}_{\mathbb{Q}}$ carry a Weil datum satisfying AW1–AW4 for $(\Lambda, \nVdash)$ that also satisfies AW5, with completion $\mathscr{H}^1$ of (V^1, h_1). Is there an analytic model Hilbert space $\mathcal{M}$ carrying the operator M_{an} of multiplication by z, and a unitary $U : \mathscr{H}^1 \to \mathcal{M}$ matching the cup-product and star structures, such that on complete domains

$$U\big[-i(\Theta_1 - \tfrac{1}{2})\big]U^{-1} = -M_{\mathrm{an}}, \tag{12.11}$$

and such that U preserves spectral multiplicities, trace distributions, and the normalization of the determinant in (AW-det)?

By AW5, $-i(\Theta_1 - \frac{1}{2})$ is the self-adjoint operator T_1; the minus sign in (12.11) reflects the correspondence $\rho = \frac{1}{2} + i\tau \leftrightarrow z = -\tau$ of Definition 1.16. If some zero of F is multiple, the scalar model $\mathcal{K}(\vartheta)$ of Theorem 11.25 cannot serve as $\mathcal{M}$, since its operator has simple spectrum (Proposition 9.8); one must then prove simplicity separately, weaken the conclusion to zero location (Question 12.14), or construct and justify additional multiplicity data. An isomorphism of uncompleted vector spaces, a comparison valid only after assuming RH, or an agreement of integer-twist cones (Proposition 9.3) does not give (12.11).

13. Open obligations

The following questions are what remains between the conditional Theorem 6.3 and an unconditional statement, together with one question on the trace-formula axiom (Question 13.3). They are listed in a useful order of attack; none is resolved in these notes.

Question 13.1. Define a functor on $\overline{\mathrm{Sch}}_1$ (Definition 1.19), or on the single object $\overline{X}_{\mathbb{Q}}$, together with its complex coefficient structure, spectral core, and completion, without presupposing its spectrum. Does it pass the Galois-functoriality test of Remark 9.21, that is, is it genuinely complex or quaternionic rather than the complexification of a real theory?

Question 13.2. Is there an arithmetic object whose trace formula or norm identity is exactly the explicit formula, with prime terms, gamma terms, and normalization derived from the object? In particular, is there an independently defined transform satisfying the norm identity of Question 12.14 for every test function (Proposition 12.9)?

The trace-formula axiom for $\overline{X}_K$ is Definition 5.9; by Theorem 7.43 it determines the degree-one spectrum on every tail class of test functions, by Theorem 7.50 not on a one-sided bounded window, and by Proposition 7.52 a two-sided bounded window detects every spectrum added on the critical line. What remains is the following.

Question 13.3. Let K be a number field such that every zero of ξ_K has real part $\frac{1}{2}$, let $L > 0$, and let $\nu = \sum_\rho \mathrm{ord}_\rho(\xi_K)\,\delta_{\mathrm{Im}\,\rho}$. Is there a measure $\mu = \sum_\gamma m(\gamma)\delta_\gamma \neq \nu$ on $\mathbb{R}$, with integer weights $m(\gamma) \geq 0$, $m(-\gamma) = m(\gamma)$, $m(0)$ even, and polynomially bounded counting function, such that the Fourier transform of $\mu - \nu$ vanishes on $(-L, L)$ in the sense of distributions? Equivalently: is there a Weil datum on $\overline{X}_K$ satisfying AW1, AW3, AW4, AW5, and AW6′ on $\mathcal{S}_{(-L,L)}$ whose degree-one spectrum is not $\mathrm{div}\,\xi_K$?

By Proposition 7.52 such a μ cannot satisfy $\mu \geq \nu$: it must omit some zeros. The trivial model with spectrum $\{\frac{1}{2} + i\gamma\}$, weighted by m, gives the equivalence, as in the proof of Theorem 7.50. By analogy with the extension problem for positive definite functions on an interval, which is in general not uniquely solvable, one expects the answer to be yes; no such μ is constructed here.

Question 13.4. Can the positive compatible form of AW4, or the global invertibility and endpoint condition of Theorem 11.27, be established? Algebraic duality, a positive Hamiltonian on an initial interval, and semilocal compression do not suffice (Propositions 6.11, 9.12, and 9.16).

Question 13.5. Is the goal RH alone, RH together with simplicity of the zeros, or the full determinant identity (AW-det) with zero orders? For the last, construct the multiplicity-preserving object required by Question 12.15; a scalar de Branges model cannot supply it when a zero is multiple (Proposition 9.8).

Question 13.6. Formulate an actual all-place descent: objects, overlap maps, topology, and derived gluing linking the finite-place realizations of Theorems 11.3, 11.5, 11.6, and 11.7 with the archimedean realization of Theorem 11.8. Determinant formulas alone are not descent data (Remark 9.27; cf. Proposition 9.26).

Question 13.7. Which additional information (analytically controlled dependence on the twist, determinant metrics, a resolvent-level comparison) allows the fixed-twist comparisons of Conjecture 11.15 to recover a global generator? The cones alone cannot (Proposition 9.3).

The remaining open items elsewhere in the notes are Question 12.10 (a repaired transform in Suzuki's framework) and Questions 12.14 and 12.15 (the two comparison targets). The class of zeta functions is fixed in Section 7; the self-contained account of Weil's positivity for curves is Section 4.3. Beyond the class $\mathcal{C}_{\mathrm{Ded}}$, the obligations above have a class-level form: a functor on $\overline{\mathrm{Sch}}_1$ with Galois descent (Definition 7.10), whose construction would also yield Artin's conjecture for rational characters (Proposition 7.12). The Künneth conjecture (Conjecture 1.40) is a further open item: it asks for the arithmetic surface on which the Hodge index clause of Theorem 8.34 could be proved.

14. Software and AI assistance

14.1. **Methods: typesetting.** The manuscript was assembled from AI research reports and a Lean formalization, then revised into a survey of the previous work described in Section 2. The drafting passes of 25–26 September 2026 used Anthropic's Claude Opus 5.5, model identifier `claude-opus-5-5[1m]`, in Claude Code 2.1.282 [Ant26a, Ant26c, product description], with the `using-latexpowers` router and `harvest-dumps` workflow. They selected and organized the reports into definitions, statements, and proofs, checked exact constants and source locators, and revised the text after independent agent reviews. The starting reports were AI-generated; the research questions and the request to prepare these notes came from Taylor Dupuy. The human author has only thumbed through the manuscript, as stated in the abstract.

14.2. **Methods: agent management.** The repository has draft contracts describing its agent entry point, file layout, canonical TEX files, writing conventions, prior decisions, and AI disclosure. Those proposals remain unadopted. The recorded synthesis used `using-latexpowers` and `math-workflow` to organize the writing, `create-exposition` and `rapid-prototype` to prepare its plan, `fill-in-prototype` for drafting, and `contradiction-check` to compare conflicting claims. A disagreement about the determinant normalization was recorded and investigated rather than silently resolved by choosing one report. The models identify the origin of the drafts; they are neither authors nor mathematical authorities.

This manuscript has not been checked by a human. Readers use it at their own risk and are responsible for independently verifying any claim, proof, computation, or citation they rely on.

The workflow instructions named here are credited to Mathcity and its mirror of the author's agent skills [Dup26a, LaTeX workflows], [Dup26b, workflow mirror]. These are later reference snapshots, not evidence of the versions executed in September. The exact historical versions and public retrievability of all corresponding pinned files remain incompletely verified; the procedural descriptions here do not require those files.

14.3. **Methods: record keeping.** The project retains provenance and token records and the dated research, review, and verification packages supporting them. The provenance record states the prompts, requested and observed models, harnesses, work performed, derivations, searches, and negative findings; `update-ai-usage` is its writer. The token record, maintained by `update-tokens`, distinguishes measured counts, estimates, unavailable counts, and prices. The available counts are tabulated in Appendix A. Missing runtime or token information is

reported as missing. In particular, the requested model for the report of 11 September 2026 was `gpt-6-astra`, but the actual runtime model and effort were not exposed; that request does not establish what ran. Some citation helpers in Claude Code were recorded only as `sonnet`; their exact model version was not retained.

14.4. **Methods: natural language proofs.** Research questions were sent through `frontier-dump`, with the report distinguishing conditional arguments, counterexamples, computations, and open obligations. The report of 19 September 2026 used OpenAI's `gpt-6-astra` in the Codex desktop harness at reasoning effort `ultra` [Ope26a, Ope26c, product description]; the earlier report of 11 September requested the same model and effort, with the runtime gap just stated. The reports on zeta-function axioms, finite-field calibration, the arithmetic Künneth conjecture, determinant normalization, and the trace-formula axiom on 25 September 2026, and on the Eilenberg–Moore comparison on 26 September 2026, used Claude Fable 5.1, model identifier `claude-fable-5-1`, in Claude Code 2.1.282 [Ant26a, Ant26b, product description]. Their reasoning setting was not exposed. The exposition re-derived the proofs and corrected claims in those reports; a research report was an input, never mathematical warrant. The four-question prompt, supplied proposal, research report, and caller validation for the September search are retained in `ai/2026-09-19-arithmetic-weil-cohomology/`. The preceding investigation is retained in `ai/2026-09-11-prove-riemann-hypothesis/`; its workflow was then named `astra-dump`. The source-search records of 26 September are in the two `provenance-a` and `provenance-b` packages under `ai/`. These records distinguish supplied ideas, generated candidates, source checks, and later corrections. They support the account in Section 1.1; they are not measurements of a constructed arithmetic cohomology or an experiment establishing RH.

14.5. **Methods: adversarial reviews.** The recorded drafting process used independent agents with `fp-finder-latex` and `critical-review` for outline, presentation, and integrated-manuscript reviews, followed by `referee-report` on successive completed drafts and revisions through `revise`. Some reviewers shared the writer's model family. The review records distinguish those passes from self-checks, list findings and closure tests, and record what was corrected or deferred. These are simulated reviews by agents; they establish no human review or acceptance of the mathematics.

14.6. **Methods: AI assisted computations.** The research and review packages contain re-runnable checks with exact rational arithmetic and numerical diagnostics. Recorded systems include Python 3.12.14 and 3.13.7 [Pyt26, project description], mpmath 1.3.0 [The26b, documentation], SymPy 1.13.2 [MSP+17, software description], and NumPy 2.3.2 [HMvdW+20, pp. 357–362]; several runs used the Python runtime from SageMath 10.8 [The26c, project website]. They checked finite-dimensional witnesses, finite-field point counts, determinant constants, explicit-formula identities, and signs in the de Branges examples. These numerical diagnostics are not interval certificates and do not support an existence claim for the conjectural cohomology. Source checks opened the cited papers when obtainable; an inaccessible original is cited together with an opened source reporting it. The manuscript was built with `latexmk` [Col26, package description]; the historical build version is not recorded.

14.7. **Methods: AI statement.** This statement was revised by an OpenAI GPT-6 agent in the Codex desktop harness on 14 October 2026 [Ope26a, product description]. Its exact runtime model identifier and reasoning effort were not exposed to this drafting agent. The `write-materials-and-methods` and `latex-ai-statement` instructions were consulted to prepare this revision; the final disclosure was also checked against the recorded trail. The structure follows the software statement of the author's PCF manuscript. The requested models were

`gpt-6-astra` at reasoning effort `high` for the source and mathematical review, `gpt-6.1-sol` at effort `high` for the disclosure comparison, and `gpt-6-luna` at effort `medium` for the independent `check-paragraph` reviewers, all in Codex desktop. Their exact runtime identifiers were not independently reported. The continuation of this revision reconstructed the literature-search purpose from the retained reports and made the source category a parameter of the common spectral formalism. It requested `gpt-6-astra` at `high` for source and mathematical review, `gpt-6.1-sol` at `high` for archival provenance and paragraph coordination, and `gpt-6-luna` at `medium` for fresh reviews of changed paragraphs. Actual runtime identifiers were not independently reported. This was an archival reconstruction and manuscript revision, not a new `frontier-dump` investigation. The paragraph verdicts are retained separately from the manuscript.

Human review and acceptance are not established by these agent passes. Historical workflow versions were not recorded for several passes; public, version-pinned sources for those usages have not been established.

On 15 October 2026 an OpenAI `gpt-6-sol` agent at `xhigh` effort in the Codex desktop harness [Ope26d, model description] drafted Definition 10.1, Conjecture 10.2, and the revised opening [Ope26a, product description]. These are observed harness labels, not a provider response identity. The inputs were the existing manuscript and the AI-generated research report in `ai/2026-10-15-single-deninger-formalism/`. The agent checked the relevant Flach–Morin and CCM source statements and made two revisions after a `gpt-6-luna` agent review; another `gpt-6-luna` agent audited the writing advice and a third checked the revised opening [Ope26b, model description]. The sources motivate the two separate comparisons but do not state their common-realization conjecture. The bounded literature check found no existing combined comparison; it was not exhaustive. These agent reviews are not human verification, and neither comparison has been proved.

On 17 October 2026, a Codex desktop `gpt-6-sol` agent at `xhigh` effort used `harvest-dumps` to add Subsection 2.2 and the corresponding negative register entries. Its selected input was the AI-generated research package `ai/2026-10-17-standard-conjectures-bsd/`, prepared by an observed `gpt-6-astra` agent at `ultra` effort through `frontier-dump` [Ope26a, product description], [Ope26d, model description], and [Ope26c, model description]. The workflow files came from a local `agent-skills` checkout at revision `8ed41877`; `harvest-dumps` had uncommitted local changes. A public [Dup26b, workflow reference snapshot] describes the workflow family, but the exact executed files have no verified public snapshot. The package distinguished Grothendieck's geometric standard conjectures from Tate's divisor conjecture and from arithmetic Abel–Jacobi and special-value comparisons. The conditional rank argument and the function-field implication are expositions of Deninger's proposal and the cited arithmetic theorems. Independent Codex agents reviewed the outline, candidate mathematics and presentation; these reviews and the source audit are retained in the dated `scratch/` and `ai/` packages. No human verification or mathematical acceptance is established by these agent checks.

14.8. **Proofreading and adversarial review.** The Opus drafting passes and Opus or Fable reviewer passes of 25–26 September 2026 used the harnesses stated above. Reviews caught errors in determinant normalization, Galois descent, tensor-product hypotheses, the strength of the Hodge-index clause, the scope of a compact-support argument, and citation locators. The later Eilenberg–Moore harvest replaced a weight argument by a degree argument and supplied a direct proof where a cochain enhancement had previously been assumed without proof. No human referee has reviewed the manuscript.

14.9. **Literature review.** Source searches were run in the research packages and source retrieval through `track-down-reference` checked pinpoint locators. On 26 September 2026 two

Claude Fable 5.1 agents in Claude Code, with delegated subagents, searched for prior sources of the theorem-class statements. They used arXiv, general web search, publisher and bibliographic databases, and a local corpus of the cited literature. The Semantic Scholar index was rate-limited and could not be queried systematically. The resulting prior-work survey, provenance table, and attributions were written by Claude Opus 5.5 in Claude Code. Source verification by these agents is not verification by the human author. Some earlier packages had recorded unperformed literature searches; those are gaps, not zero-hit searches.

14.10. **AI findings and procedure.** The inputs were the author's research questions, the published work surveyed in Section 2, and the earlier AI-generated reports. The negative findings are retained in Section 9 and the register of Section 9.5; they delimit the claims made by the axioms and candidate constructions. The principal corrections and contributions from the drafting and research passes are:

- The descent argument was restricted to what permutation characters actually determine (Proposition 7.12); a claimed determinant normalization was replaced by the computation of Theorem 7.37.
- The tensor-square analysis acquired the zero-counting hypothesis in Theorem 8.12(iii), the sign condition in Proposition 8.15(iii), and the one-way implication distinguished in Proposition 8.36. The unsupported compact-support injectivity claim was not used in Proposition 8.22.
- The trace-formula report's prime-sum divergence was placed in the spectral normalization in Proposition 7.53; the derivation after Theorem 7.41 uses a polynomial bound on horizontal segments in place of the cited estimate.
- The exposition gathers Deninger's conditional positivity argument, the finite-field model, and the counterexamples, with their prior sources and the method comparisons recorded in Subsection 14.11.
- The later cochain comparison was harvested into Section 8, with the corrections described under "Proofreading and adversarial review."

These drafting and research contributions were produced by the Opus, Fable, and Codex passes dated and identified above. Their proofs and the cited mathematical sources, rather than a model's assertions, supply the arguments.

14.11. **Provenance.** Section 2 and Table 10 give the sources and relations for the statements retained in these notes. In particular, the implication from a positive star compatible with the flow to the Riemann hypothesis is Deninger's formal argument after Serre, rather than a new discovery of the agents. The table distinguishes imported, classical, adapted, and standard statements from those for which the 26 September search found no prior appearance. The last class asserts only that search outcome, with the limits just stated; it asserts no novelty. Some individual earlier proof-writing and Lean tasks recorded no separate literature search, a limitation that the later statement survey does not turn into an exhaustive novelty check.

Table 10 lists the theorem-class statements of Sections 3–13 with the place where each appears in the literature or where its idea comes from. The classes are: *imported*, a result stated with its source and locator; *classical*, a result in the literature, cited at the statement; *adapted*, a variant or reformulation of a result in the literature; *standard*, textbook material or folklore; and *not found*, a statement for which no prior appearance was found in the literature we searched. The last class is not a claim of novelty: the search used arXiv, general web search, and a local corpus of the cited literature, the Semantic Scholar index could not be queried systematically, and the statements concerned are mostly explicit examples and elementary equivalences of a kind that may well appear in print. Definitions are listed only where they carry an attribution.

Table 10: Provenance of the statements of these notes.

Statement	Where it appears, or where the idea comes from	Class
Section 3		
Def. 1.7, Prop. 3.3	[Ser60, pp. 392–393]; [Den94, (7.11)–(7.13), pp. 32–33]; [Den98, p. 169]; [Den23, 2.7, p. 3]	classical
Def. 1.1	[FM20, Definition 2.1(i), p. 521]; [Den94, §1]; the accumulation clause added	adapted
Thms. 3.4, 3.5, 3.8	sources cited at the statements	imported
Props. 3.1, 3.6	consequences of Hecke's theorem and of $\eta(s) > 0$ on $(0, 1)$	standard
Def. 3.9, Prop. 3.10	Weil's functional [Wei72, §16]; [Bom06, §V, p. 9]; [Bom00, Theorem 1, p. 191]	classical
Section 4		
Def. 4.1	hypothesis of [Ser60, Théorème 1]	adapted
Props. 4.2, 4.8	[Ser60, p. 393]; [Kle68, Lemma 4.6, p. 385]; [Mil15, Proposition 3.10]	classical
Props. 4.3, 4.4	linear algebra	standard
Prop. 4.6	that positivity is needed: [Kle68, p. 360]; the example itself	not found
Thms. 4.9, 4.10	sources cited at the statements	imported
Prop. 4.11	Weil [Wei41]; [Ser60, Théorème 2, p. 394]; [Kle68, Lemma 4.6 and Theorem 4.7]; [Mil15, Theorem 1.27, p. 21]	classical
Prop. 4.12	[Den10, p. 3]	standard
Defs. 4.14, 4.15, Prop. 4.16(i)–(ii)	[Den94, §2, Corollary (2.6) and (2.8), and §4, p. 16]; [Hes18, Remark 7.2]	classical
Prop. 4.16(iii)–(iv), Prop. 4.17	[Den10, p. 4] (integrated and infinitesimal star laws)	adapted
Prop. 4.19(i)	[Kle68, Lemma 4.6]	classical
Prop. 4.19(ii)–(iii)	ingredients [Kle68, Lemma 4.6]	not found
Prop. 4.20	Weil; [Mil08, §14]; [Kle68, Lemma 4.6]	adapted
Section 5		
Defs. 5.2, 5.3	[Den94, (1.1)–(1.7), pp. 3–6]; [FM20, Definition 2.1]	imported
Def. 1.3	[Den98, pp. 167–168]	classical
Defs. 1.4, 5.5, 5.6	[Den10, Conjecture 2, p. 3, and pp. 4–6]; [FM20, Conjecture 2.3, p. 522]; [Den98, p. 168, p. 172]; [Den94, (7.13), (7.19)]; [Kle68, Lemma 4.6]	adapted
Prop. 1.6	derivation, trace, and positivity in the former weighted internal axioms; proved here	adapted
Defs. 5.7, 5.8, 5.9	Weil, as reported in [Bom06, §V]; [Den98, (4) and (10), pp. 170–172]; [Den01, (17)–(19), pp. 15–16]	adapted
Section 6		
Prop. 6.1	[Den10, p. 5]; [Den94, (7.12)–(7.13)]	classical
Prop. 6.2(i)–(ii)	[Kle68, Lemma 4.6] in additive form; [Den94, (7.13)]	classical
Prop. 6.2(iii)	linear algebra	not found
Thm. 6.3	Deninger's formal argument [Den94, (7.11)–(7.13)], [Den98, pp. 168–169], [Den10, pp. 4–5], [Den23, 2.7], [FM20, Conjecture 2.3(iv)–(v)], after [Ser60]	classical
Prop. 6.5	spectral theorem; [Den94, p. 33]	standard
Prop. 6.6, Rem. 6.7	Hilbert–Pólya: [Con99, p. 3], [Den92, p. 147], [Sar04, p. 5]	standard
Prop. 6.9	indefinite forms; [Den10, pp. 7–9]	standard
Props. 6.11, 6.12	mechanism [Den23, 2.7]; the example itself	not found
Section 7		
Thms. 7.2, 7.3	sources cited at the statements	imported

Statement	Where it appears, or where the idea comes from	Class
Thm. 7.4	(i), (iv) as Thm. 6.3; (ii), (iii) Artin formalism	classical
Props. 7.6, 7.7	[Den23, 2.6–2.9, pp. 3–4]; [Den25, Proposition 3.1]	adapted
Prop. 7.9	[Den23, 2.9 and Theorem 2.13]	adapted
Def. 7.10	[Den23, p. 4]; [Den25, §§2.8–2.10]	adapted
Prop. 7.11	Artin; [Ser77, §13.1, Theorems 29–30]	classical
Prop. 7.12	ingredients [Ser77, §13.1, p. 104], [Den23, 2.10–2.12]	not found
Prop. 7.14	[Den25, Theorem 2.15]	adapted
Prop. 7.15	[Den23, Theorem 2.13]	imported
Prop. 7.16	[Den23, 2.7–2.8]; [Den25, Proposition 3.1]	adapted
Prop. 7.17, Def. 7.18	[Den94, §4, pp. 16–17]	classical
Thms. 7.19, 7.34	sources cited at the statements	imported
Thm. 7.20, Prop. 7.21	[Den94, (2.7)–(2.10), (4.1)]; [Hes18, Proposition 7.1]	classical
Props. 7.22, 7.51	direct computation	standard
Prop. 7.23	[Den98, pp. 167–168, Proposition 3.1]	classical
Thm. 7.24	(i)–(iii) [Den94, §4, (4.1)], [Den98, Theorem 3.3, p. 172]; (iv)–(vi) Weil, [Kle68, Lemma 4.6], [Mil15, §3]	classical
Prop. 7.27	[Den10, Conjecture 1(c), p. 2]	classical
Thm. 7.28	(i) [Den98, p. 172], [Den10, p. 5]; (ii)–(iii) stated here	adapted
Prop. 7.30	the example itself	not found
Rem. 7.31	[Den10, pp. 6–7]; [Den94, (7.11)–(7.13)]	adapted
Thm. 7.32	[Den92, Theorem 3.3 and Remark 3.4(1)]; Soulé; [Kur88, p. 24]	imported
Thm. 7.37	[Den94, Theorem 7.18(ii)] (Soulé, Illies); [Kur88, p. 24]; the constant relative to Λ_K not found written out	classical
Thm. 7.38, Prop. 7.39	[Den94, Lemma (1.7), Theorem (7.20), and (7.19.1)]; the explicit residues not found	adapted
Prop. 7.40	compare [Den92, Remark (3.4)(2)]	not found
Props. 7.35, 7.36, 7.42, 7.46, 7.53	Stirling, beta integral, uniqueness of Dirichlet series, Stone's theorem, $\sum_p 1/p = \infty$	standard
Thm. 7.41	Weil, as reported in [Bom06, §V]; [Den01, Theorem 4.1]; [Bar81]	imported
Thm. 7.43	trace formula as an expectation: [Den98, (10)], [Den01, (19)]; the equivalence	not found
Thm. 7.45	(i) Grothendieck; (ii) [Den02, (2), p. 3], [Den98, (10)]; (iii)	classical
Thm. 7.47	[Den94, p. 33], [Den98, p. 169]	adapted
Prop. 7.49	Weil; [Bom06, §V, p. 9]	classical
Thm. 7.50, Prop. 7.52	the examples themselves	not found
Section 8		
Props. 8.2, 8.3, 8.19, 8.22, 8.29, 8.32	linear algebra, Kronecker sums, Euler characteristics, the Künneth formula over a field	standard
Def. 1.38	[Kle68, §1.2, p. 363, and p. 360]	adapted
Def. 1.39	[Man95, §1.6, p. 133]	adapted
Def. 8.4	[Kur92, p. 219]; [Man95, §5.2, p. 158]	adapted
Prop. 8.5	$\operatorname{Spec}\mathbb{Z} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Z} = \operatorname{Spec}\mathbb{Z}$: [Man95, p. 121]; [Den94, p. 41]	adapted
Props. 8.6, 8.9, 8.30	ingredients [Mil13, pp. 22–23, 89–90, 105]; the examples themselves	not found
Prop. 8.8, Rem. 8.10	[McC01, Theorem 7.6, p. 241]	adapted
Thm. 8.12, Props. 8.16, 8.17	[Man95, p. 132]; [Den94, (7.32), p. 42]	adapted

Statement	Where it appears, or where the idea comes from	Class
Prop. 8.13	the cone of [Kur92, p. 219]; the statement	not found
Prop. 8.15	tensor sums of self-adjoint operators	standard
Prop. 8.18	[Den98, pp. 167–168, p. 181]; [Kur92, p. 220]	adapted
Prop. 8.20	[Ser70, §3, (25)]; [Neu99, VII.12, p. 535]	classical
Rem. 8.21	[Den98, p. 184]; Serre [Mil15, fn. 25, p. 27]	imported
Conj. 1.40	[Den94, (7.32)]; [CC19, §3]; [Sch18, §6]; the form stated	not found
Thm. 8.23	sources cited in the proof	adapted
Def. 8.24, Props. 8.27, 8.28	[Lur11, Definition 1.1.7, Proposition 1.1.9, Corollary 1.1.10]; Dwyer	adapted
Prop. 8.25	[Sta26, Tag 0F0C] 0F0C, [Sta26, Tag 0F0G] 0F0G	imported
Prop. 8.31	bar resolution [Lan63, X.2, Theorem 2.1], [Wei94, 8.6.12]	adapted
Prop. 8.33	[Kah18, Proposition 2.2.3]	classical
Thm. 8.34	[Kle68, 3.11]; [Mum74, §21, Theorem 1]; [Ser60, Théorème 2]	adapted
Props. 8.36, 8.37	compare [Den10, pp. 4–5]	not found
Section 9		
Props. 9.1, 9.12, 9.20	compare [Lag05, Lemma 2.1], [Suz12, Theorem 2.3], [Saa11, Theorem 3.9, p. 63]	not found
Prop. 9.3	[Den91, §1, pp. 246–247]; [QHS93, (2), p. 215]	adapted
Props. 9.4, 9.6, 9.10, 9.26	Hasse's bound; closed symmetric operators; linear algebra; [Den92, (1.3), (2.3)]	standard
Prop. 9.8	[Lag06a, p. 4 and Theorem 1(ii)]	adapted
Prop. 9.14	[dB50, §7(b), p. 223]	standard
Prop. 9.16	[Yos92, Theorem 1, p. 310]; [Bom00, Theorem 12]; [Bur00]	standard
Prop. 9.22	Serre [Mil15, (2.2), p. 27]; [Den23, §1, p. 1]	classical
Rems. 9.21, 9.24, 9.27, 9.28, 9.30	sources cited at the remarks	imported
Prop. 9.32	[Rie59, p. 4]; [dB50, (7.2)–(7.3), p. 220]	classical
Prop. 9.34, Rems. 9.36, 9.37	about version 3 of [Suz25]; corrected in [Suz26, p. 27]	not found
Section 11		
all statements	sources cited at the statements	imported
Section 12		
Prop. 12.1, Rem. 12.2	[Lag06a, Theorem 1 and (10), p. 9]	classical
Prop. 12.3	[Lag05, Lemma 2.1]	standard
Prop. 12.4	de Branges's theorem; [Rom14, (12)–(13) and Proposition 4]	classical
Prop. 12.7	Weil; [Bom00, Theorem 1]; [CC20, Proposition C.1]	classical
Prop. 12.9	the "if" direction of [Suz25, Theorems 1.4 and 4.4]	adapted
Question 12.10	the definition of [Suz26, p. 27]	adapted
Props. 12.12, 12.13	residual bounds [Saa11, Theorem 3.9]; Hurwitz's theorem	standard

The statements for which no prior appearance was found in the literature we searched are Propositions 4.6, 4.19(ii)–(iii), 6.2(iii), 6.11, 6.12, 7.12, 7.30, 7.40, 7.52, 8.6, 8.9, 8.13, 8.30, 8.36, 8.37, 9.1, 9.12, 9.20, and 9.34, Theorems 7.43 and 7.50, Conjecture 1.40 in the form stated, and Remarks 9.36 and 9.37.

14.12. **Due diligence.** The finite-field package tested the axioms against their classical model; the Künneth package tested stronger product expectations; the determinant package computed the normalization; the trace-formula package tested how much of the spectrum its test-function class detects; and the cochain package investigated the Eilenberg–Moore comparison. Their

corrections and counterexamples are retained with the positive statements. The recorded computations and agent reviews leave the geometric and analytic existence obligations open.

On 15 October 2026, an OpenAI `gpt-6-astra` agent at reasoning effort `max`, using the Codex desktop harness and the `frontier-dump` workflow, [Ope26a, Ope26c, product description], searched for a counterexample to Flach–Morin's Conjecture 11.15. It checked the finite-field point and the published finite-field theorem and found no verified counterexample to the full conjecture in that bounded search. Proposition 9.3 records the narrower failure of integer-twist cone data to determine a zero spectrum; it is not a counterexample to the conjecture. The search and its source checks are retained in `ai/2026-10-15-flach-morin-cyclic-assembly/`.

The same research pass attempted to rule out any assembly of finite-prime and archimedean local theories. It established the specific direct-sum obstruction in Proposition 9.26, but found no universal impossibility theorem for constructions with global relations, quotients, or other comparison maps. The class of all possible assemblies was not specified, so the unsuccessful search is not evidence that an assembly exists. This revision was integrated by an OpenAI `gpt-6-sol` agent at reasoning effort `xhigh` in Codex desktop on the same date [Ope26a, product description]. The mathematical warrants are the displayed arguments and the cited primary papers, not the agents' judgments.

A second `frontier-dump` pass on 15 October 2026 used an OpenAI `gpt-6-astra` agent at `max` effort in the Codex desktop harness [Ope26a, Ope26c]. It tested the proposed adelic positivity proof, local-to-global comparison routes, and Scholze's analytic-ring example. It obtained neither global positivity nor an arithmetic spectral realization. The compact-support argument in Proposition 9.17 and the fixed-radius argument in Proposition 10.3 are the narrower negative results incorporated here. Its primary-source checks and finite verification are recorded in `ai/2026-10-15-adelic-positivity-prismatic-scholze/`. The present manuscript integration was made by an OpenAI `gpt-6-sol` agent at `xhigh` effort in Codex desktop. The mathematical warrants remain the proofs and the cited sources.

A further `frontier-dump` pass on 15 October 2026 used an OpenAI `gpt-6-astra` agent at `max` effort in Codex desktop [Ope26a, Ope26c, product description]. It checked two function-field trace examples and searched for a comparison between Flach–Morin's integer-twist complexes and the Connes–Consani–Marcolli adelic cyclic quotient. The examples are calculations in known function-field cases; they do not establish adelic positivity over $\mathbb{Q}$. The searched sources supplied a common Frobenius spectrum in the function-field setting, but no verified canonical map between the two constructions. The methods differ: one uses motivic regulators and fixed integer twists, while the other uses an adelic convolution action on cyclic homology. The bounded search and calculations are retained in `ai/2026-10-15-ccm-two-examples-flach-morin/`. Theorem 11.20 is the direct specialization of [CCM09, Corollary 6.3, p. 29] to the trivial character of $\mathbb{Q}$; it proves no new positivity statement. The manuscript integration was made by an OpenAI `gpt-6-sol` agent at `xhigh` effort in Codex desktop. Its warrant is the cited criterion and the displayed proof.

The terminology and scope remarks 1.25 and 11.21 were drafted on 15 October 2026 by an OpenAI `gpt-6-sol` agent at `xhigh` effort in Codex desktop [Ope26a, product description]. The pass compared Flach–Morin's conjectural dynamical groups and fixed-twist Weil–Arakelov complexes with Connes–Consani–Marcolli's cyclic trace realization. A bounded search of those papers and targeted uniqueness queries found no theorem characterizing a global Deninger cohomology theory uniquely; this is a search limit, not a nonexistence result. The source checks and query record are in `scratch/2026-10-15-deninger-terminology/report.md`. The mathematical warrants are the cited papers and AW1–AW2, not the drafting agent.

The later naming pass on 15 October 2026 was drafted by an OpenAI `gpt-6-sol` agent at `xhigh` effort in Codex desktop [Ope26a, product description]. It checked Flach–Morin's name

and fixed-twist comparison, CCM's relative H^1 construction, Hesselholt's TP determinant, and Connes–Consani's ordinary cyclic archimedean determinant against the cited primary papers. The methods have different inputs: motivic regulators, an adelic cyclic quotient, a circle Tate construction on THH, and archimedean cyclic homology. The table and remarks record the correction that Hesselholt's formula uses TP rather than TC. This pass found no direct identification among the adelic quotient, TP, and TC in those sources; it does not rule out a future comparison. The source audit is in `scratch/2026-10-15-fm-ccm-names/report.md`.

14.13. **Formalization.** The Lean formalization was written on 25–26 September 2026 by Claude Opus 5.5 in Claude Code 2.1.282. It uses Lean 4.34.1 [dMU21, pp. 625–635] and mathlib at tag `v4.34.1` [The26a, project repository]; the recorded checks build the project and inspect the axioms of its declarations. The table below states claim by claim what is encoded, what is assumed, and what is outside the formalization. The finite-dimensional reductions and their proof checking do not construct the arithmetic cohomology or discharge the analytic inputs. Machine checking is not human review of the formalization choices.

The `lean/` directory of this repository contains a Lean 4 formalization (Lean 4.34.1 with Mathlib v4.34.1) of the statements listed below, by claim identifier; the tree contains no `axiom` declaration and no `sorry`, so every declaration in it depends only on the standard axioms `propext`, `Classical.choice`, and `Quot.sound`; the script `lean/scripts/PrintAxioms.lean` prints the axioms of 202 declarations, among them every theorem named in the table below (definitions such as `completedXi` and `dirichletDatum` are not among the 202). The formal statements are conditional in the same way as the text: they take the axioms as hypotheses and never assert the Riemann hypothesis.

The central formal fact is an equivalence about the structure

`WeilRH.ArithmeticWeilCohomologyQMult`,

which records a complex inner product space (the positive form h_1 of AW4), a linear map Θ that is locally finite, identity (1.1) with $j = 1$, and the order condition of AW2: every eigenspace $\ker(\Theta - \rho)$ is finite-dimensional of dimension equal to the order of ξ at ρ. The theorem `WeilRH.nonempty_arithmeticWeilCohomologyQMult_iff` states that this structure is inhabited if and only if the Mathlib statement `RiemannHypothesis` holds. This is the formal counterpart of Proposition 6.6 with multiplicities; it encodes no cup product, trace, or star, so it formalizes the spectral part of Proposition 7.7(iv) for $K = \mathbb{Q}$ only. The data encoded are equivalent to the Riemann hypothesis, and the content of a proof along these lines lies in constructing them from independent arithmetic input.

The formalization builds on, and in places overlaps with, earlier and contemporaneous work. The statement `RiemannHypothesis` is Mathlib's [TL26] `github.com/leanprover-community/mathlib4`, introduced by D. Loeffler, and the functional equations of the completed zeta function used here come from the same development. The definition of ξ as `completedXi` coincides with `Complex.riemannXi` of the PrimeNumberTheoremAnd project [Cip26] `github.com/AlexKontorovich/PrimeNumberTheoremAnd` and with `riemannXi` of Bulka's formalization of Li's criterion [Bul26] `github.com/nicholasbulka/li-criterion-rh-equivalence-lean`. An independent, contemporaneous Lean formalization of the Hilbert–Pólya reduction [Csi26] `github.com/frostynumeral/hilbert-polya-lean-verification` proves that a symmetric operator whose eigenvalues realize the zeros implies `RiemannHypothesis`, and that such a realization exists if and only if `RiemannHypothesis` holds; the structure above differs in taking identity (1.1) rather than symmetry as its hypothesis, in encoding local finiteness, and in tracking multiplicities, and the caveat that the existence of the structure is a reformulation of the

Riemann hypothesis applies to both. The thick closure `EM.ThickClosure` of the table below is a local packaging of Mathlib's `ObjectProperty.triangEnvelope` [SR26] github.com/leanprover-community/mathlib4/pull/37046, built on J. Riou's triangulated subcategories, and `EM.thickClosure_le` is the analogue of `triangEnvelope_le_iff`.

Claim (statement)	Lean declarations	Fidelity
C-PUR (Prop. 3.3)	`re_eq_half_of_hasEigenvalue`, `isSemisimple_of_additive`, `maxGenEigenspace_eq_eigenspace_of_additive`	faithful for parts (i) and (iii); the eigenvalue part holds in any complex inner product space, semisimplicity in finite dimension; part (ii) has no declaration of its own
C-WEIL (Prop. 4.2)	`norm_eq_rpow_of_weil`, `isSemisimple_of_weil`	faithful to the modulus and diagonalizability clauses, with $q > 0$ in place of $q > 1$; the h-orthonormal basis and the orthogonality of distinct eigenspaces are not stated; no ℓ-adic cohomology, geometric polarization, or embedding ι
C-RH (Thm. 6.3, $K = \mathbb{Q}$)	`ArithmeticWeilCohomologyQ.riemannHypothesis`, `ArithmeticWeilCohomologyQMult.riemannHypothesis`	AW2 enters only as a divisor condition (set-level, or with zero orders); regularized determinants, degrees 0 and 2, AW3, the star, and cup product are not encoded
C-CONV (Prop. 6.6)	`nonempty_arithmeticWeilCohomologyQ_iff`, `nonempty_arithmeticWeilCohomologyQMult_iff`	faithful; the second keeps multiplicities; the models are built from the zeros and are not arithmetic
ξ and its zeros (Def. 1.16)	`completedXi`, `completedXi_eq_zero_iff`, `analyticOrderNatAt_completedXi_pos_iff`	faithful; ξ is defined as an entire function and its zeros are exactly the nontrivial zeros for every ρ
flow form of (1.1)	`P_iff_flow`, `P_iff_flow_of_finiteDimensional`	faithful for bounded generators and finite-dimensional blocks; unbounded generators and AW5 not covered
Q1-L4.2 (Prop. 6.1)	`WeightedStarDatum.derivation_law`, `WeightedStarDatum.re_eq_of_hasEigenvalue`, `WeightedStarDatum.isSemisimple`	faithful to the weighted-slice algebraic identity, specialized to $w = 0$ in Proposition 6.1; it does not encode the full unital datum. Semisimplicity is finite-dimensional; perfectness of the pairing, graded commutativity, other degrees, and AW2 not encoded

Claim (statement)	Lean declarations	Fidelity
Q2-T1′ (Prop. 4.8, Prop. 6.2)	`PositiveTrace.norm_eq_rpow_of_mulLeft`, `PositiveTrace.isSemisimple_mulLeft`, `PositiveTrace.minpoly_squarefree`, `PositiveTrace.re_eq_half_of_mulLeft`	faithful over $k = \mathbb{C}$ for any finite-dimensional complex $*$-algebra with positive trace, including the additive form used in AW4-E; Lean's `PositiveTrace.τ_pos` asks $0 < \operatorname{Re}\tau(aa^\dagger)$ and derives realness, which is equivalent to Definition 4.7 given $\tau(a^\dagger) = \overline{\tau(a)}$; the cases $k = \mathbb{Q}, \mathbb{R}$, part (ii), and the packaging of AW4-E as a subalgebra of $\mathrm{End}(V^j)$ are not encoded
C-A (Thm. 7.4)	`grh_dedekind_of_awc`, `riemannHypothesis_of_awc_dedekind_rat`, `dirichletDatum`, `grh_dirichlet_of_awc`	part (i) in class form, with the encoding of row C-RH (set-level spectrum; no cup product, trace, star, or multiplicities), which asks less of each object than AW1–AW4; Mathlib has no continuation of ζ_K, so Λ_K is supplied as data carrying Hecke's theorem, proved unique, and constructed only for $K = \mathbb{Q}$, where the statement recovers `RiemannHypothesis`; for other K the hypothesis cannot be instantiated inside this development, since no continuation is constructed in Lean, although one exists mathematically. The hypothesis quantifies over all fields K in the universe `Type` and over every continuation datum, which is harmless, since the conclusion does not depend on the continuation (`dedekindDatum_grh_iff`). Dirichlet L-functions are stated directly, not derived from part (i); parts (ii)–(iii) (class field theory, Brauer induction), descent, and multiplicities are not encoded
Q1 Cor. 5.2 (Prop. 7.7)	`WeightedStarDatum.lineModel`, `R2.counterexample`	witnesses only: a positive one-dimensional model, and the indefinite form satisfying (1.1) with an off-line eigenvalue; the latter is not repackaged as a Weil datum, and parts (i)–(iv) are not formalized

Claim (statement)	Lean declarations	Fidelity
C-B (Thm. 7.24(iv), Props. 4.16, 4.17)	`suspension_P`, `flow_suspension`, `weil_of_P_of_flow`, `suspension_P_iff_weil`	for a given positive form and, one $n \in \mathbb{Z}$ at a time, the sheet $\alpha \mapsto (\operatorname{Log}\alpha + 2\pi i n)/\log q$ with the same n on every eigenspace (Def. 4.18; Log the principal branch): the form is a positive Frobenius polarization of weight j if and only if it satisfies (1.1) for the generator of the sheet and the flow of that generator at time $\log q$ is F; the AW4-E leg of Thm. 7.24(iv), the existence direction of Prop. 4.3(i), the full direct sum, the nilpotent part, the regularized determinant (Thm. 7.20), the suspended duality and star, and ℓ-adic cohomology are not encoded
R2 (Props. 6.11, 6.12)	`R2.counterexample`, `R2.Jplus_not_commute`	faithful to the matrices, for all real $a \neq 0$ and T, including that the indefinite form satisfies (1.1); the signature $(2,2)$ of part (ii) enters only through $H^2 = I$ and $\operatorname{tr} H = 0$, and part (iii) only as one eigenvalue off the line; the graded algebra and Prop. 6.12(iii) not encoded
R3 (Prop. 9.3)	`R2.det_eq`, `R2.isUnit_sub_intCast`	the polynomial and the invertibility of integer twists, for $T \neq 0$ (the text assumes $T > 0$); "$p(s) > 0$ for real s" is formalized as $p(s) \neq 0$ for real s (`R2.p_real_ne_zero`); determinant lines not encoded

Claim (statement)	Lean declarations	Fidelity
C-K6 (Thm. 8.34(i), (iii), (iv))	`posDef_kronecker_iff`, `SurfaceDatum.h_eq_zero_of_re_ne_half`, `SurfaceDatum.hodgeIndex_iff_pureDefinite`, `SurfaceDatum.re_eq_half_of_hodgeIndex`, `SurfaceDatum.isSemisimple_of_hodgeIndex`	a finite-dimensional, blockwise Gram-matrix model of N_S (the Hodge index clause as positivity of the Kronecker products of the Gram matrices of h_1 on V_ρ and $V_{1-\rho}$); the three `SurfaceDatum` statements about the Hodge index clause assume V^1 finite-dimensional (`FiniteDimensional`), which loses nothing, because $h_\otimes$ is positive definite on the algebraic direct sum N_K if and only if it is so on each block: off-line blocks are $h_\otimes$-isotropic by part (i), and distinct on-line blocks are $h_\otimes$-orthogonal, as in the proof of part (iii); `h_eq_zero_of_re_ne_half` holds without it, and `posDef_kronecker_iff` concerns finite matrices on nonempty index sets; of part (iv) only diagonalizability and $\operatorname{Re}\rho = \frac{1}{2}$ are formalized, the clauses on AW4-V, AW4-E, and on ξ_K having no declaration; the form B is assumed nondegenerate in the first variable only (`SurfaceDatum.B_nondeg`), which is weaker than AW3 and suffices; no completion, no diagonal classes, and no Künneth map κ; the signature formula of part (ii) is not formalized, and the Riemann hypothesis for ζ_K is not derived, since that needs AW2
R12 (Prop. 9.14)	`R12.fourier_eq`, `R12.cos_add_cosh_eq_zero_plus`, `R12.cos_add_cosh_eq_zero_minus`	the atomic measure: its Fourier transform, and that $\pi \pm ia$ are zeros; that the zeros are exactly the points of part (i), simple, and non-real, is not formalized; the Gaussian variant not encoded
Def. 4.18, Prop. 4.19	`AW4EFDatum`, `SatisfiesAW4EF`, `IsSheet`, `AW4EFDatum.sheetElem`; (i): `AW4EFDatum.isUnit`, `.mul_inv`, `.isSemisimple`, `.norm_eq`; (ii): `.star_sheetElem`, `.ι_sheetElem_apply`; (iii): `.truncation`, `AW4EAddDatum.re_eq_half`; (iv): `AW4EFDatum.dual`	the subalgebra A is encoded as an abstract algebra with an injective unital map to $\mathrm{End}_{\mathbb{C}}(V)$, an equivalent formulation; the suspension $(\widehat{V}, \Theta)$, the sheet spaces, and the sheet maps are not built, so (ii) is encoded as the action of θ_σ by $\sigma(\alpha)$ on $\ker(\mathrm{F} - \alpha)$, and (iii) for $\widehat{V}'$ the sum of the copies of V indexed by finitely many sheets, without the claim that such spaces exhaust $\widehat{V}$

Claim (statement)	Lean declarations	Fidelity
Prop. 4.20	`rosati_aw4EF`, `rosati_gram_posDef`, `rosati_trace_pos`	part (iii) for an abstract algebra with a positive trace involution acting faithfully on V and an element π with $\pi\pi^\dagger = q$, all three taken as hypotheses; of part (ii) only the positivity step is formalized: a rational symmetric Gram matrix positive on $\mathbb{Q}^n$ gives a positive definite Hermitian form on $\mathbb{C}^n$; the Jacobian, $\mathrm{End}^0(J)$, $V_\ell J$, part (i), and Theorem 4.10 are not encoded
Prop. 7.51	`sum_phantomSet_pow_eq_zero`, `sum_phantomSet_pow_self`, `norm_of_mem_phantomSet`, `div_mem_phantomSet`, `spectrum_suspension_phantom`	both parts; in part (ii) the spectrum is encoded as the union of the progressions $\Sigma_q(\alpha)$, $\alpha \in S_N$, which is the index set of Definition 4.15
Thm. 7.45	`Lefschetz.altCount_eq_of_lefschetzNumber_eq`, `Lefschetz.zetaRat_eq_of_lefschetzNumber_eq`	a finite analogue of part (iii): for finitely many eigenvalues with multiplicities in each degree, the numbers N_k for all $k \geq 1$ determine the alternating multiplicities at every $\alpha \neq 0$ and the rational function $\prod_j \det(1 - \mathrm{F}T \mid V^j)^{(-1)^{j+1}}$; N_k is the alternating sum of power sums of eigenvalues, not a trace of a linear map; parts (i) and (ii) and the infinite suspended spectrum are not formalized
Props. 8.27, 8.28	`EM.ThickClosure`, `EM.thickClosure_le`, `EM.isIso_app_of_mem_thickClosure`, `EM.isIso_comp_of_mem_thickClosure`	the thick-subcategory induction in any pretriangulated category: a natural transformation of triangulated functors compatible with the shift that is an isomorphism on a class of objects is one on its thick closure, the smallest strictly full triangulated subcategory closed under direct summands containing it; étale derived categories, cochain enhancements, μ_S, Proposition 8.25, and the filtration and truncation steps of Proposition 8.28 are inputs, not formalized

All declaration names lie in the namespace `WeilRH`. The Lean conventions differ from the text in one point that is proved harmless: Mathlib's inner product is linear in the second argument, and the formal statements use the form $h(u, v) = \langle v, u\rangle$, for which identity (1.1) with real j is equivalent in either convention. The remaining formal obligations are the regularized determinants and AW-det, the continuation of Λ_K in Mathlib, parts (ii)–(iii) of Theorem 7.4, the Hilbert realization AW5, Proposition 9.6, the full suspension of Section 4.4, and the determinant and trace-formula statements of Sections 7.7 and 7.8. The file `lean/README.md`

records, for each declaration, the correspondence with the statement it formalizes, and the script `lean/scripts/PrintTypes.lean` prints its full type.

Appendix A. Token accounting

Table 12 reports the retained counting windows and estimates, not a complete bill or a total for the project. The measured 19 September window records 653,852 uncached input tokens, 16,242,944 cache-read tokens, and 107,121 output tokens, with no recorded cache writes. The historical rate-card check on 19 September 2026 used USD 10, 1, and 50 per million tokens in those three classes for the Standard scenario and 2.5 times those rates for the Fast scenario. The resulting valuations are USD 28.137514 and 70.343785, respectively. They are alternative token-charge scenarios, not amounts to add, and exclude search and other tool fees; the actual service tier and billing agreement are not recorded.

All remaining quantities below are unsplit harness aggregates or context-budget estimates. They exclude unreported primary-agent use and, where noted in the records, later review rounds. They are left unpriced here because their input, output, and cache classes cannot be recovered. Approximation signs distinguish budget estimates from retained aggregate counters; neither is a complete project total. The model identifiers and harness versions are given in Section 14.

Table 12: Retained AI usage; incomplete windows and estimates.

Task	Model and harness	Tokens	USD valuation
19 September arithmetic cohomology research	GPT-6 Astra, Codex desktop	17,003,917; class split above	28.137514 Standard; 70.343785 Fast
Finite-field calibration	Fable 5.1, Claude Code	$\approx 380,000$ unsplit	unpriced
Arithmetic Künneth research	Fable 5.1, Claude Code	$\approx 480,000$ unsplit	unpriced
Trace-formula research	Fable 5.1, Claude Code	$\approx 400,000$ unsplit	unpriced
First Lean reduction	Opus 5.5, Claude Code	$\approx 170,000$ plus 71,191	unpriced
Lean multiplicities	Opus 5.5, Claude Code	$\approx 165,000$	unpriced
First synthesis, eight descendants	Opus 5.5, Claude Code	1,996,417	unpriced
Lean class calibration and surface	Opus 5.5, Claude Code	$\approx 265,000$ and $\approx 215,000$	unpriced
Second synthesis, five descendants	Opus/Fable/Sonnet, Claude Code	1,288,264	unpriced
First revision, four descendants	Opus/Fable/Sonnet, Claude Code	1,173,021	unpriced
Determinant source-retrieval descendant	Fable 5.1, Claude Code	180,421	unpriced
Second referee pass, two descendants	Fable 5.1, Claude Code	646,091	unpriced
Third referee pass, one descendant	Fable 5.1, Claude Code	136,153	unpriced
Second revision, three descendants	Opus/Fable, Claude Code	801,948	unpriced
Fourth referee pass, two descendants	Fable 5.1, Claude Code	418,967	unpriced
Fourth revision, reviewer	Opus 5.5, Claude Code	218,300	unpriced
Eilenberg–Moore source retrieval	Fable 5.1, Claude Code	209,948	unpriced
Fifth revision, reviewer	Fable 5.1, Claude Code	192,601	unpriced

Task	Model and harness	Tokens	USD valuation
Other primary-agent work and 11 September research	Models as disclosed in Section 14	not recorded	not recorded
14 October revision and reviews	GPT-6 family; requested Astra, Sol, and Luna; Codex desktop	not exposed	not computed

References

[Ant26a] Anthropic. Claude Code, version 2.1.282, 2026. Agentic coding harness; used 2026-09-25. URL: https://claude.com/claude-code.

[Ant26b] Anthropic. Claude Fable 5.1 (model `claude-fable-5-1`), 2026. Large language model; used 2026-09-25. URL: https://www.anthropic.com.

[Ant26c] Anthropic. Claude Opus 5.5 (model `claude-opus-5-5[1m]`), 2026. Large language model; used 2026-09-25. URL: https://www.anthropic.com.

[Ant26d] Anthropic. formal-math / zeta23: Weil's explicit formula and the two-thirds theorem in Lean 4, 2026. GitHub repository `anthropics/formal-math`, subproject `zeta23`, commit fbdc36b (5 September 2026). URL: https://github.com/anthropics/formal-math.

[Ayo] Joseph Ayoub. Homepage and list of papers. Accessed 2026-09-25. URL: https://user.math.uzh.ch/ayoub/.

[Ayo18] Joseph Ayoub. Topologie feuilletée et la conservativité des réalisations classiques en caractéristique nulle, 2018. Author manuscript; see the author's homepage for its status. URL: https://user.math.uzh.ch/ayoub/PDF-Files/Feui-ConCon.pdf.

[Ayo19] Joseph Ayoub. Topologie feuilletée et théorie de Galois différentielle, 2019. Author manuscript. URL: https://user.math.uzh.ch/ayoub/PDF-Files/Feui-DGal.pdf.

[Ayo20] Joseph Ayoub. Nouvelles cohomologies de Weil en caractéristique positive. *Algebra & Number Theory*, 14(7):1747–1790, 2020. URL: https://msp.org/ant/2020/14-7/ant-v14-n7-p03-s.pdf, doi:10.2140/ant.2020.14.1747.

[Bar81] Klaus Barner. On A. Weil's explicit formula. *Journal für die reine und angewandte Mathematik*, 323:139–152, 1981. doi:10.1515/crll.1981.323.139.

[BK99] Michael V. Berry and Jonathan P. Keating. The Riemann zeros and eigenvalue asymptotics. *SIAM Rev.*, 41(2):236–266, 1999. URL: https://michaelberryphysics.files.wordpress.com/2013/07/berry307.pdf, doi:10.1137/S0036144598347497.

[BM17] Andrew J. Blumberg and Michael A. Mandell. The strong Künneth theorem for topological periodic cyclic homology, 2017. Locators refer to arXiv:1706.06846v2. URL: https://arxiv.org/abs/1706.06846, arXiv:1706.06846.

[BMS18] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. Integral p-adic Hodge theory. *Publications Mathématiques de l'IHÉS*, 128:219–397, 2018. Locators refer to the author's preprint. URL: https://people.mpim-bonn.mpg.de/scholze/integralpadicHodge.pdf, arXiv:1602.03148.

[BMS19] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. Topological Hochschild homology and integral p-adic Hodge theory. *Publications Mathématiques de l'IHÉS*, 129:199–310, 2019. Locators refer to the author's preprint. URL: https://people.mpim-bonn.mpg.de/scholze/bms2.pdf, arXiv:1802.03261.

[Bom00] Enrico Bombieri. Remarks on Weil's quadratic functional in the theory of prime numbers, I. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, 11(3):183–233, 2000. URL: http://www.bdim.eu/item?id=RLIN_2000_9_11_3_183_0.

[Bom06] Enrico Bombieri. The Riemann hypothesis. In *The Millennium Prize Problems*, pages 107–124. Clay Mathematics Institute, Cambridge, MA, 2006. Official problem description (2000); locators refer to the claymath.org PDF. URL: https://www.claymath.org/wp-content/uploads/2022/05/riemann.pdf.

[Bor09] James Borger. Λ-rings and the field with one element, 2009. arXiv:0906.3146v1; locators refer to this version. URL: https://arxiv.org/abs/0906.3146, arXiv:0906.3146.

[BS22] Bhargav Bhatt and Peter Scholze. Prisms and prismatic cohomology. *Annals of Mathematics*, 196(3):1135–1275, 2022. Locators refer to the author's preprint. URL: https://people.mpim-bonn.mpg.de/scholze/prisms.pdf, arXiv:1905.08229.

[Bul26] Nicholas Bulka. li-criterion-rh-equivalence-lean, 2026. GitHub repository `nicholasbulka/li-criterion-rh-equivalence-lean`, commit 35df682 (4 September 2026). URL: https://github.com/nicholasbulka/li-criterion-rh-equivalence-lean.

[Bur00] Jean-François Burnol. Sur les formules explicites I: analyse invariante. *C. R. Acad. Sci. Paris Sér. I Math.*, 331(6):423–428, 2000. Locators refer to arXiv:math/0101068. arXiv:math/0101068, doi:10.1016/S0764-4442(00)01687-6.

[CC09] Alain Connes and Caterina Consani. Schemes over $\mathbb{F}_1$ and zeta functions, 2009. arXiv:0903.2024v3; published in Compos. Math., doi:10.1112/S0010437X09004692. URL: https://arxiv.org/abs/0903.2024, arXiv:0903.2024.

[CC14a] Alain Connes and Caterina Consani. The arithmetic site. *C. R. Math. Acad. Sci. Paris*, 352(12):971–975, 2014. Locators refer to arXiv:1405.4527v1. URL: https://arxiv.org/abs/1405.4527v1, arXiv:1405.4527.

[CC14b] Alain Connes and Caterina Consani. Cyclic homology, Serre's local factors and the λ-operations. *Journal of K-Theory*, 14:1–45, 2014. Locators refer to arXiv:1211.4239v1. URL: https://arxiv.org/abs/1211.4239v1, arXiv:1211.4239, doi:10.1017/IS014006012JKT270.

[CC15] Alain Connes and Caterina Consani. The scaling site. *C. R. Math. Acad. Sci. Paris*, 353(12):1–6, 2015. Locators refer to arXiv:1507.05818v2. URL: https://arxiv.org/abs/1507.05818v2, arXiv:1507.05818.

[CC16] Alain Connes and Caterina Consani. Geometry of the arithmetic site. *Adv. Math.*, 291:274–329, 2016. Locators refer to arXiv:1502.05580v1. URL: https://arxiv.org/abs/1502.05580v1, arXiv:1502.05580, doi:10.1016/j.aim.2015.11.045.

[CC17] Alain Connes and Caterina Consani. Geometry of the scaling site. *Selecta Mathematica. New Series*, 23(3):1803–1850, 2017. Locators refer to arXiv:1603.03191v1. URL: https://arxiv.org/abs/1603.03191v1, arXiv:1603.03191.

[CC19] Alain Connes and Caterina Consani. The Riemann–Roch strategy, complex lift of the scaling site. In *Advances in Noncommutative Geometry*. Springer, 2019. Locators refer to arXiv:1805.10501v1. URL: https://arxiv.org/abs/1805.10501v1, arXiv:1805.10501.

[CC20] Alain Connes and Caterina Consani. Weil positivity and trace formula, the archimedean place, 2020. Locators refer to arXiv:2006.13771v1. URL: https://arxiv.org/abs/2006.13771v1, arXiv:2006.13771.

[CC23] Alain Connes and Caterina Consani. Riemann–Roch for $\overline{\operatorname{spec} F}$. *Bull. Sci. Math.*, 187:103293, 2023. Locators refer to arXiv:2205.01391v2. URL: https://arxiv.org/abs/2205.01391v2, arXiv:2205.01391, doi:10.1016/j.bulsci.2023.103293.

[CCM09] Alain Connes, Caterina Consani, and Matilde Marcolli. The Weil proof and the geometry of the adeles class space. In *Algebra, Arithmetic, and Geometry: In Honor of Yu. I. Manin, Vol. I*, volume 269 of *Progr. Math.*, pages 339–405. Birkhäuser, Boston, 2009. Locators refer to arXiv:math/0703392v1. URL: https://arxiv.org/abs/math/0703392v1, arXiv:math/0703392, doi:10.1007/978-0-8176-4745-2_8.

[CCM24] Alain Connes, Caterina Consani, and Henri Moscovici. Zeta zeros and prolate wave operators: semilocal adelic operators. *Annals of Functional Analysis*, 15(4):38, 2024. Locators refer to arXiv:2310.18423v2. URL: https://arxiv.org/abs/2310.18423v2, arXiv:2310.18423, doi:10.1007/s43034-024-00388-z.

[CCM25] Alain Connes, Caterina Consani, and Henri Moscovici. Zeta spectral triples, 2025. Locators refer to arXiv:2511.22755v1. URL: https://arxiv.org/abs/2511.22755v1, arXiv:2511.22755.

[Cip26] Matteo Cipollina. `Complex.riemannXi` and Hadamard factorization in the PrimeNumberTheoremAnd project, 2026. GitHub repository `AlexKontorovich/PrimeNumberTheoremAnd`, file `CompletedXi.lean` (PR #1503, 9 June 2026). URL: https://github.com/AlexKontorovich/PrimeNumberTheoremAnd.

[CL00] J. Brian Conrey and Xian-Jin Li. A note on some positivity conditions related to zeta and L-functions. *International Mathematics Research Notices*, 2000(18):929–940, 2000. Locators refer to arXiv:math/9812166v1. URL: https://arxiv.org/abs/math/9812166v1, arXiv:math/9812166, doi:10.1155/S1073792800000489.

[Cog14] James W. Cogdell. On Artin L-functions, 2014. Expository manuscript; locators refer to the author's posted PDF. URL: https://people.math.osu.edu/cogdell.1/artin-www.pdf.

[Col26] John Collins. latexmk: Fully automated LaTeX document generation, 2026. Official CTAN package page, accessed 14 October 2026; historical manuscript-build version not recorded. URL: https://ctan.org/pkg/latexmk.

[Con] Brian Conrad. Math 210B. Frobenius–Schur indicator. Course handout, Stanford University. URL: http://math.stanford.edu/~conrad/210BPage/handouts/realrep.pdf.

[Con99] Alain Connes. Trace formula in noncommutative geometry and the zeros of the Riemann zeta function. *Selecta Math. (N.S.)*, 5(1):29–106, 1999. Locators refer to arXiv:math/9811068v1. URL: https://arxiv.org/abs/math/9811068v1, arXiv:math/9811068.

[Con00] Alain Connes. Noncommutative geometry and the Riemann zeta function. In *Mathematics: Frontiers and Perspectives*, pages 35–54. Amer. Math. Soc., Providence, RI, 2000. URL: https://www.asc.ohio-state.edu/math/events/connes/zeta2.pdf.

[Con03] J. Brian Conrey. The Riemann hypothesis. *Notices Amer. Math. Soc.*, 50(3):341–353, 2003. URL: https://www.ams.org/notices/200303/fea-conrey-web.pdf.

[Con16] Alain Connes. An essay on the Riemann hypothesis. In Jr. John F. Nash and Michael Th. Rassias, editors, *Open Problems in Mathematics*, pages 225–257. Springer, 2016. Locators refer to arXiv:1509.05576v1. URL: https://arxiv.org/abs/1509.05576v1, arXiv:1509.05576.

[Cou83] Jean Cougnard. Les travaux de A. Fröhlich, Ph. Cassou-Noguès et M. J. Taylor sur les bases normales. *Astérisque*, 105–106:25–38, 1983. Séminaire Bourbaki, 35e année, exp. no. 598. URL: http://www.numdam.org/item?id=SB_1982-1983__25__25_0.

[Csi26] Ádám János Csillag. hilbert-polya-lean-verification: a machine-checked conditional Hilbert–Pólya reduction, 2026. GitHub repository `frostynumeral/hilbert-polya-lean-verification`, commit 1838675 (24 September 2026). URL: https://github.com/frostynumeral/hilbert-polya-lean-verification.

[dB50] N. G. de Bruijn. The roots of trigonometric integrals. *Duke Math. J.*, 17(3):197–226, 1950. doi:10.1215/S0012-7094-50-01720-0.

[dB86] Louis de Branges. The Riemann hypothesis for Hilbert spaces of entire functions. *Bull. Amer. Math. Soc. (N.S.)*, 15(1):1–17, 1986.

[dB92] Louis de Branges. The convergence of Euler products. *J. Funct. Anal.*, 107(1):122–210, 1992. doi:10.1016/0022-1236(92)90103-P.

[Dei04] Anton Deitmar. Schemes over $\mathbb{F}_1$, 2004. arXiv:math/0404185v7. URL: https://arxiv.org/abs/math/0404185, arXiv:math/0404185.

[Del74] Pierre Deligne. La conjecture de Weil. I. *Publications Mathématiques de l'IHÉS*, 43:273–307, 1974. URL: http://www.numdam.org/item?id=PMIHES_1974__43__273_0.

[Del80] Pierre Deligne. La conjecture de Weil. II. *Publ. Math. Inst. Hautes Études Sci.*, 52:137–252, 1980. URL: http://www.numdam.org/item?id=PMIHES_1980__52__137_0.

[Den91] Christopher Deninger. On the Γ-factors attached to motives. *Invent. Math.*, 104(2):245–261, 1991. doi:10.1007/BF01245075.

[Den92] Christopher Deninger. Local L-factors of motives and regularized determinants. *Invent. Math.*, 107(1):135–150, 1992. doi:10.1007/BF01231885.

[Den94] Christopher Deninger. Motivic L-functions and regularized determinants. In *Motives (Seattle, WA, 1991)*, volume 55, Part 1 of *Proc. Sympos. Pure Math.*, pages 707–743. Amer. Math. Soc., Providence, RI, 1994. Locators refer to the page numbering of the author's typescript (pp. 1–46); venue data not printed in that copy.

[Den98] Christopher Deninger. Some analogies between number theory and dynamical systems on foliated spaces. *Doc. Math.*, Extra Volume ICM 1998, I:163–186, 1998.

[Den01] Christopher Deninger. Number theory and dynamical systems on foliated spaces. *Jahresber. Deutsch. Math.-Verein.*, 103(3):79–100, 2001. Locators refer to arXiv:math/0204110v1. URL: https://arxiv.org/abs/math/0204110, arXiv:math/0204110.

[Den02] Christopher Deninger. On the nature of the "explicit formulas" in analytic number theory—a simple example. In *Number Theoretic Methods (Iizuka, 2001)*, volume 8 of *Dev. Math.*, pages 97–118. Kluwer Acad. Publ., 2002. Locators refer to arXiv:math/0204194v1. URL: https://arxiv.org/abs/math/0204194, arXiv:math/0204194.

[Den08] Christopher Deninger. Analysis on foliated spaces and arithmetic geometry. In Katrin Tent, editor, *Groups and Analysis: The Legacy of Hermann Weyl*, volume 354 of *London Math. Soc. Lecture Note Ser.*, pages 174–190. Cambridge Univ. Press, 2008. arXiv:0709.2801, under the title "Analogies between analysis on foliated spaces and arithmetic geometry"; locators refer to arXiv version 1. arXiv:0709.2801, doi:10.1017/CBO9780511721410.010.

[Den10] Christopher Deninger. The Hilbert–Polya strategy and height pairings, 2010. Preprint, arXiv:1001.1621v1; locators refer to this version. URL: https://arxiv.org/abs/1001.1621v1, arXiv:1001.1621.

[Den23] Christopher Deninger. There is no "Weil-"cohomology theory with real coefficients for arithmetic curves, 2023. arXiv:2204.02714v2, final version; published in Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 25 (2024), no. 3, 1717–1725, doi:10.2422/2036-2145.202204_005. Locators refer to arXiv version 2. URL: https://arxiv.org/abs/2204.02714v2, arXiv:2204.02714.

[Den25] Christopher Deninger. Is there a Birch and Swinnerton-Dyer conjecture for Dedekind zeta functions?, 2025. arXiv:2504.15767v3 (1 April 2026); locators refer to this version. arXiv:2504.15767.

[dMU21] Leonardo de Moura and Sebastian Ullrich. The Lean 4 theorem prover and programming language. In *Automated Deduction—CADE 28*, volume 12699 of *Lecture Notes in Computer Science*, pages 625–635. Springer, 2021. Project toolchain: Lean 4.34.1. doi:10.1007/978-3-030-79876-5_37.

[DS01] Christopher Deninger and Wilhelm Singhof. A counterexample to smooth leafwise Hodge decomposition for general foliations and to a type of dynamical trace formulas. *Annales de l'Institut Fourier*, 51(1):209–219, 2001. URL: https://www.numdam.org/item/10.5802/aif.1821/, doi:10.5802/aif.1821.

[Dup17] Taylor Dupuy. Hadamard's theorem and entire functions of finite order, 2017. Course notes, Math 331. URL: https://tdupu.github.io/complexspring2017/hadamard.pdf.

[Dup26a] Taylor Dupuy. mathcity: mathematical research workflows and policies, 2026. Reference snapshot 0205e0adfb8963a843c28616fcedceafc7ca5680 (2 October 2026), checked in the local Git history. This later reference version does not identify the versions executed during the September 2026 manuscript preparation. URL: https://github.com/tdupu/mathcity.

[Dup26b] Taylor Dupuy. mathcity: mirrored agent workflow skills, 2026. Reference snapshot c87fb990708ba1e8493133c6a8f17877004444c1 (4 October 2026), docs/agent-skills-mirror.md; mirrors agent-skills revision 76230b62375beedaaf4cc7bf2f058f2bf357c409. Later reference version, not the September execution version. URL: https://github.com/tdupu/mathcity.

[Dur07] Nikolai Durov. New approach to Arakelov geometry, 2007. arXiv:0704.2030v1; locators refer to this version. URL: https://arxiv.org/abs/0704.2030, arXiv:0704.2030.

[Dwy74] William G. Dwyer. Strong convergence of the Eilenberg–Moore spectral sequence. *Topology*, 13:255–265, 1974.

[Fla04] Matthias Flach. The equivariant Tamagawa number conjecture: a survey, 2004. Contemporary Mathematics 358, 79–125; locators refer to the author's manuscript. URL: https://virtualmath1.stanford.edu/~conrad/BSDseminar/refs/equivariant.pdf.

[FM18] Matthias Flach and Baptiste Morin. Weil-étale cohomology and zeta-values of proper regular arithmetic schemes. *Doc. Math.*, 23:1425–1560, 2018. arXiv:1605.01277.

[FM20] Matthias Flach and Baptiste Morin. Deninger's conjectures and Weil–Arakelov cohomology. *Münster Journal of Mathematics*, 13:519–540, 2020. URL: https://d-nb.info/1216560145/34, doi:10.17879/90169642993.

[Gro69] Alexander Grothendieck. Standard conjectures on algebraic cycles. In *Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968)*, pages 193–199. Oxford Univ. Press, London, 1969.

[Har77] Robin Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1977.

[Hes18] Lars Hesselholt. Topological Hochschild homology and the Hasse–Weil zeta function. In *An Alpine Bouquet of Algebraic Topology*, volume 708 of *Contemporary Mathematics*, pages 157–180. American Mathematical Society, Providence, RI, 2018. Locators refer to arXiv:1602.01980v2. URL: https://arxiv.org/abs/1602.01980v2, arXiv:1602.01980, doi:10.1090/conm/708/14264.

[Hin10] Marc Hindry. Introduction to zeta and L-functions from arithmetic geometry and some applications. Mini-curso, XXI Escola de Álgebra, Brasília, 2010. Lecture notes, version of August 2010; locators refer to the page numbering of that copy.

[HMvdW+20] Charles R. Harris, K. Jarrod Millman, Stéfan J. van der Walt, et al. Array programming with NumPy. *Nature*, 585(7825):357–362, 2020. Recorded project version: 2.3.2. doi:10.1038/s41586-020-2649-2.

[HSW21] Elchin Hasanalizade, Quanli Shen, and Peng-Jie Wong. Counting zeros of Dedekind zeta functions, 2021. Locators refer to arXiv:2102.04663v3. URL: https://arxiv.org/abs/2102.04663, arXiv:2102.04663.

[Kah18] Bruno Kahn. *Fonctions zêta et L de variétés et de motifs*. Calvage & Mounet, 2018. arXiv:1512.09250; locators refer to the arXiv version.

[Kle68] Steven L. Kleiman. Algebraic cycles and the Weil conjectures. In *Dix exposés sur la cohomologie des schémas*, pages 359–386. North-Holland, Amsterdam, 1968.

[Kle94] Steven L. Kleiman. The standard conjectures. In *Motives (Seattle, WA, 1991)*, volume 55, Part 1 of *Proc. Sympos. Pure Math.*, pages 3–20. Amer. Math. Soc., Providence, RI, 1994.

[Kra05] János Kramár. Artin's and Brauer's theorems on induced characters, 2005. Course handout, University of Toronto, MAT445. URL: https://www.math.utoronto.ca/fiona/courses/mat445/artinbrauer.pdf.

[KT03] Kazuya Kato and Fabien Trihan. On the conjectures of Birch and Swinnerton-Dyer in characteristic $p > 0$. *Inventiones Mathematicae*, 153:537–592, 2003. doi:10.1007/s00222-003-0299-2.

[Kur88] Nobushige Kurokawa. Parabolic components of zeta functions. *Proc. Japan Acad. Ser. A Math. Sci.*, 64(1):21–24, 1988.

[Kur92] Nobushige Kurokawa. Multiple zeta functions: an example. In Nobushige Kurokawa and Toshikazu Sunada, editors, *Zeta Functions in Geometry (Tokyo, 1990)*, volume 21 of *Advanced Studies in Pure Mathematics*, pages 219–226. Kinokuniya; Mathematical Society of Japan, Tokyo, 1992. doi:10.2969/aspm/02110219.

[Lag99] Jeffrey C. Lagarias. On a positivity property of the Riemann ξ-function. *Acta Arith.*, 89(3):217–234, 1999. Correction: Acta Arith. 116 (2005), no. 3, 293–294. doi:10.4064/aa-89-3-217-234.

[Lag05] Jeffrey C. Lagarias. Zero spacing distributions for differenced L-functions. *Acta Arith.*, 120(2):159–184, 2005. Locators refer to arXiv:math/0601653v2. URL: https://arxiv.org/abs/math/0601653v2, arXiv:math/0601653.

[Lag06a] Jeffrey C. Lagarias. Hilbert spaces of entire functions and Dirichlet L-functions. In Pierre Cartier, Bernard Julia, Pierre Moussa, and Pierre Vanhove, editors, *Frontiers in Number Theory, Physics, and Geometry I*, pages 365–377. Springer, Berlin, 2006. Pages as in zbMATH and Crossref; the author's publication list gives pp. 367–379. Locators refer to the author's PDF. URL: https://websites.umich.edu/~lagarias/doc/debranges-houches.pdf, doi:10.1007/978-3-540-31347-2_10.

[Lag06b] Jeffrey C. Lagarias. The Riemann hypothesis: arithmetic and geometry. In Nigel Higson and John Roe, editors, *Surveys in Noncommutative Geometry*, volume 6 of *Clay Mathematics Proceedings*, pages 127–141. Amer. Math. Soc.; Clay Math. Inst., Providence, RI, 2006. Locators refer to the author's revision dated 4 May 2007. URL: https://websites.umich.edu/~lagarias/doc/mt-holyoke-rev.pdf.

[Lan63] Saunders Mac Lane. *Homology*, volume 114 of *Grundlehren Math. Wiss.* Springer, 1963.

[Lor11] Oliver Lorscheid. The geometry of blueprints. Part I: Algebraic background and scheme theory, 2011. arXiv:1103.1745v2. URL: https://arxiv.org/abs/1103.1745, arXiv:1103.1745.

[Lor18] Oliver Lorscheid. $\mathbb{F}_1$ for everyone, 2018. URL: https://arxiv.org/abs/1801.05337, arXiv:1801.05337.

[LRS22] Stephen Lichtenbaum, Niranjan Ramachandran, and Takashi Suzuki. The conjectures of Artin–Tate and Birch–Swinnerton-Dyer. *Épijournal de Géométrie Algébrique*, 6, 2022. Locators refer to arXiv:2101.10222v5. URL: https://arxiv.org/abs/2101.10222v5, arXiv:2101.10222, doi:10.46298/epiga.2022.7482.

[Lur11] Jacob Lurie. Derived algebraic geometry XIII: Rational and p-adic homotopy theory, 2011. Preprint dated 15 December 2011. URL: https://people.math.harvard.edu/~lurie/papers/DAG-XIII.pdf.

[Man95] Yuri I. Manin. Lectures on zeta functions and motives (according to Deninger and Kurokawa). *Astérisque*, 228:121–163, 1995. URL: http://www.numdam.org/item?id=AST_1995__228__121_0.

[MC26] Math Inc (Gauss autoformalization agent) and Dean Cureton. Riemannhypothesiscurves: the Riemann hypothesis for hyperelliptic curves over finite fields in Lean 4, 2026. GitHub repository `math-inc/RiemannHypothesisCurves`, commit c99880d. URL: https://github.com/math-inc/RiemannHypothesisCurves.

[McC01] John McCleary. *A User's Guide to Spectral Sequences*, volume 58 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2001.

[Mey05] Ralf Meyer. A spectral interpretation for the zeros of the Riemann zeta function. In *Mathematisches Institut, Georg-August-Universität Göttingen: Seminars Winter Term 2004/2005*, pages 117–137. Universitätsdrucke Göttingen, Göttingen, 2005. Locators refer to arXiv:math/0412277. URL: https://arxiv.org/abs/math/0412277, arXiv:math/0412277.

[Mil08] James S. Milne. Abelian varieties (v2.00), 2008. Course notes, 166+vi pages. URL: https://www.jmilne.org/math/CourseNotes/AV.pdf.

[Mil13] James S. Milne. Lectures on étale cohomology (v2.21), 2013. Course notes, 202 pages. URL: https://www.jmilne.org/math/CourseNotes/LEC.pdf.

[Mil15] James S. Milne. The Riemann hypothesis over finite fields: from Weil to the present day, 2015. Contribution to *The legacy of Bernhard Riemann after one hundred and fifty years*, Adv. Lect. Math. 35, vol. II (2016), 487–565; reprinted in ICCM Not. 4 (2016), no. 2, 14–52. Locators refer to arXiv:1509.00797v1 (2015). URL: https://arxiv.org/abs/1509.00797, arXiv:1509.00797.

[Mil20] James S. Milne. Class field theory (v4.03), 2020. Course notes; locators refer to the printed page numbers of the Introduction. URL: https://www.jmilne.org/math/CourseNotes/CFT.pdf.

[MSP+17] Aaron Meurer, Christopher P. Smith, Mateusz Paprocki, et al. SymPy: symbolic computing in Python. *PeerJ Computer Science*, 3:e103, 2017. Recorded project version: 1.13.2. doi:10.7717/peerj-cs.103.

[Mum74] David Mumford. *Abelian Varieties*, volume 5 of *Tata Institute of Fundamental Research Studies in Mathematics*. Oxford University Press, London, 2 edition, 1974. First edition 1970; corrected reprint, Hindustan Book Agency, 2008.

[Neu99] Jürgen Neukirch. *Algebraic Number Theory*, volume 322 of *Grundlehren Math. Wiss.* Springer, 1999.

[Ope26a] OpenAI. Codex desktop, 2026. Agent harness; used 2026-09-11, 2026-09-19, and 2026-10-15. URL: https://openai.com/codex.

[Ope26b] OpenAI. GPT-6 luna (model `gpt-6-luna`), 2026. Large language model; used 2026-10-15. URL: https://developers.openai.com/api/docs/models/gpt-6-luna.

[Ope26c] OpenAI. GPT-6 (model `gpt-6-astra`), 2026. Large language model; used 2026-09-11 and 2026-09-19. URL: https://openai.com.

[Ope26d] OpenAI. GPT-6 sol (model `gpt-6-sol`), 2026. Large language model; used 2026-10-15. URL: https://developers.openai.com/api/docs/models/gpt-6-sol.

[PL09] Javier López Peña and Oliver Lorscheid. Mapping $\mathbb{F}_1$-land: an overview of geometries over the field with one element, 2009. arXiv:0909.0069v1. URL: https://arxiv.org/abs/0909.0069, arXiv:0909.0069.

[PM18] Ricardo Pérez-Marco. Notes on the Riemann hypothesis, 2018. Locators refer to arXiv:1707.01770v2. URL: https://arxiv.org/abs/1707.01770v2, arXiv:1707.01770.

[Pyt26] Python Software Foundation. Python, 2026. Official project website, accessed 14 October 2026. Recorded project runtimes: Python 3.12.14 and 3.13.7. URL: https://www.python.org/about/.

[QHS93] J. R. Quine, S. H. Heydari, and R. Y. Song. Zeta regularized products. *Trans. Amer. Math. Soc.*, 338(1):213–231, 1993.

[Rie59] Bernhard Riemann. On the number of prime numbers less than a given quantity (Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse), 1859. Monatsberichte der Berliner Akademie, November 1859. English translation by David R. Wilkins, preliminary version, December 1998. URL: https://www.maths.tcd.ie/pub/HistMath/People/Riemann/Zeta/EZeta.pdf.

[Rom14] Roman Romanov. Canonical systems and de Branges spaces, 2014. Locators refer to arXiv:1408.6022v1. URL: https://arxiv.org/abs/1408.6022v1, arXiv:1408.6022.

[Saa11] Yousef Saad. *Numerical Methods for Large Eigenvalue Problems*, volume 66 of *Classics in Applied Mathematics*. SIAM, Philadelphia, revised edition, 2011.

[Sar04] Peter Sarnak. Problems of the Millennium: the Riemann hypothesis (2004). Clay Mathematics Institute Annual Report 2004, 2004. URL: https://www.claymath.org/library/annual_report/xSarnak_RH.pdf.

[Sch18] Peter Scholze. p-adic geometry, 2018. Proceedings of the International Congress of Mathematicians (Rio de Janeiro, 2018); locators refer to the author PDF. URL: https://people.mpim-bonn.mpg.de/scholze/Rio.pdf, arXiv:1712.03708.

[Sch26a] Peter Scholze. Berkovich motives, 2026. arXiv:2412.03382v3 (January 2026), final version; published in J. Amer. Math. Soc. 39 (2026), no. 3, 697–764, doi:10.1090/jams/1068. Locators refer to the version 3 PDF. URL: https://people.mpim-bonn.mpg.de/scholze/BerkovichMotives.pdf, arXiv:2412.03382.

[Sch26b] Peter Scholze. Lectures on analytic geometry, 2026. Locators refer to arXiv:2605.03655v1. URL: https://arxiv.org/abs/2605.03655v1, arXiv:2605.03655.

[Ser60] Jean-Pierre Serre. Analogues kählériens de certaines conjectures de Weil. *Ann. of Math. (2)*, 71(2):392–394, 1960. Extrait d'une lettre à A. Weil, 9 novembre 1959. URL: https://www.jstor.org/stable/1970088, doi:10.2307/1970088.

[Ser77] Jean-Pierre Serre. *Linear Representations of Finite Groups*, volume 42 of *Graduate Texts in Mathematics*. Springer, New York, 1977. Translated by L. L. Scott. doi:10.1007/978-1-4684-9458-7.

[Ser70] Jean-Pierre Serre. Facteurs locaux des fonctions zêta des variétés algébriques (définitions et conjectures). *Sém. Delange–Pisot–Poitou*, 11(2), 1969/70. Exposé 19. URL: http://www.numdam.org/item/SDPP_1969-1970__11_2_A4_0/.

[SH11] Dániel Schumayer and David A. W. Hutchinson. Physics of the Riemann hypothesis. *Rev. Mod. Phys.*, 83(2):307–330, 2011. Locators refer to arXiv:1101.3116v1. URL: https://arxiv.org/abs/1101.3116v1, arXiv:1101.3116.

[Sil09] Joseph H. Silverman. *The Arithmetic of Elliptic Curves*, volume 106 of *Graduate Texts in Mathematics*. Springer, Dordrecht, 2 edition, 2009. doi:10.1007/978-0-387-09494-6.

[Sou03] Christophe Soulé. Les variétés sur le corps à un élément, 2003. arXiv:math/0304444v1. URL: https://arxiv.org/abs/math/0304444, arXiv:math/0304444.

[SR26] Justus Springer and Joël Riou. `ObjectProperty.triangEnvelope` and `ObjectProperty.IsTriangulated` (mathlib 4), 2026. Files `Mathlib/CategoryTheory/Triangulated/Generators.lean` (J. Springer, PR #37046, 2026) and `Subcategory.lean` (J. Riou, PR #11740, 2024), at tag v4.34.1. URL: https://github.com/leanprover-community/mathlib4/pull/37046.

[SS94] Martin Schröter and Christophe Soulé. On a result of Deninger concerning Riemann's zeta function. In *Motives (Seattle, WA, 1991)*, volume 55, Part 1 of *Proc. Sympos. Pure Math.*, pages 745–747. Amer. Math. Soc., Providence, RI, 1994. doi:10.1090/pspum/055.1/1265548.

[Sta26] The Stacks Project, 2026. Tags 03QP, 0F0C and 0F0G consulted 26 September 2026. URL: https://stacks.math.columbia.edu.

[Sut17a] Andrew V. Sutherland. 18.785 number theory I, Lecture #17: The functional equation, 2017. MIT lecture notes, Fall 2017; Theorem 17.17. URL: https://math.mit.edu/classes/18.785/2017fa/LectureNotes17.pdf.

[Sut17b] Andrew V. Sutherland. 18.785 number theory I, Lecture #19, 2017. MIT lecture notes, Fall 2017; Remark 19.13 (Hecke's functional equation, stated without proof). URL: https://math.mit.edu/classes/18.785/2017fa/LectureNotes19.pdf.

[Suz12] Masatoshi Suzuki. A canonical system of differential equations arising from the Riemann zeta-function, 2012. Locators refer to arXiv:1204.1827v2. URL: https://arxiv.org/abs/1204.1827v2, arXiv:1204.1827.

[Suz21] Masatoshi Suzuki. Hamiltonians arising from L-functions in the Selberg class. *Journal of Functional Analysis*, 281(8):109116, 2021. Locators refer to arXiv:1606.05726v3. URL: https://arxiv.org/abs/1606.05726v3, arXiv:1606.05726, doi:10.1016/j.jfa.2021.109116.

[Suz25] Masatoshi Suzuki. On the Hilbert space derived from the Weil distribution. *Canadian Journal of Mathematics*, 2025. Locators refer to arXiv:2301.00421v3. URL: https://arxiv.org/abs/2301.00421v3, arXiv:2301.00421, doi:10.4153/S0008414X25101739.

[Suz26] Masatoshi Suzuki. Weil's quadratic form via the screw function, 2026. Locators refer to arXiv:2606.09096v3 (23 September 2026). URL: https://arxiv.org/abs/2606.09096v3, arXiv:2606.09096.

[The26a] The mathlib Community. mathlib4: the Lean mathematical library, 2026. Project dependency at tag v4.34.1, commit d13f23b723b8a846827a245b89c10fc7d3f11612, recorded in lean/lake-manifest.json. URL: https://github.com/leanprover-community/mathlib4.

[The26b] The mpmath development team. *mpmath 1.3.0 documentation*, 2026. Version 1.3.0; official documentation, accessed 14 October 2026. URL: https://mpmath.readthedocs.io/en/1.3.0/.

[The26c] The Sage Developers. *SageMath, the Sage Mathematics Software System*, 2026. Version 10.8 as recorded in the project run logs; citation format from the official SageMath FAQ, accessed 14 October 2026. URL: https://www.sagemath.org/.

[TL26] The Mathlib Community and David Loeffler. `RiemannHypothesis` in `Mathlib/NumberTheory/LSeries/RiemannZeta.lean`, 2026. File `Mathlib/NumberTheory/LSeries/RiemannZeta.lean` at tag v4.34.1; the definition was introduced in mathlib3 PR #19131 (D. Loeffler, 2023), ported in PR #5556 and restated in PR #13273. URL: https://github.com/leanprover-community/mathlib4/blob/d13f23b723b8a846827a245b89c10fc7d3f11612/Mathlib/NumberTheory/LSeries/RiemannZeta.lean.

[TV05] Bertrand Toën and Michel Vaquié. Au-dessous de Spec $\mathbb{Z}$, 2005. arXiv:math/0509684v4. URL: https://arxiv.org/abs/math/0509684, arXiv:math/0509684.

[Ulm12] Douglas Ulmer. CRM lectures on curves and Jacobians over function fields, 2012. Locators refer to arXiv:1203.5573v2. URL: https://arxiv.org/abs/1203.5573v2, arXiv:1203.5573.

[Vez10] Alberto Vezzani. Deitmar's versus Toën–Vaquié's schemes over $\mathbb{F}_1$, 2010. arXiv:1005.0287v2; locators refer to this version. URL: https://arxiv.org/abs/1005.0287, arXiv:1005.0287.

[Wei40] André Weil. Sur les fonctions algébriques à corps de constantes fini. *C. R. Acad. Sci. Paris*, 210:592–594, 1940.

[Wei41] André Weil. On the Riemann hypothesis in function-fields. *Proc. Nat. Acad. Sci. U.S.A.*, 27(7):345–347, 1941. doi:10.1073/pnas.27.7.345.

[Wei48a] André Weil. *Sur les courbes algébriques et les variétés qui s'en déduisent*. Number 1041 in Actualités Sci. Ind. Hermann, Paris, 1948. Publ. Inst. Math. Univ. Strasbourg 7 (1945).

[Wei48b] André Weil. *Variétés abéliennes et courbes algébriques*. Number 1064 in Actualités Sci. Ind. Hermann, Paris, 1948. Publ. Inst. Math. Univ. Strasbourg 8 (1946).

[Wei49] André Weil. Numbers of solutions of equations in finite fields. *Bulletin of the American Mathematical Society*, 55(5):497–508, 1949. URL: https://projecteuclid.org/journals/bulletin-of-the-american-mathematical-society/volume-55/issue-5/Numbers-of-solutions-of-equations-in-finite-fields/bams/1183513798.full, doi:10.1090/S0002-9904-1949-09219-4.

[Wei52] André Weil. Sur les "formules explicites" de la théorie des nombres premiers. In *Communications du Séminaire Mathématique de l'Université de Lund [Medd. Lunds Univ. Mat. Sem.], Tome Supplémentaire (dédié à M. Riesz)*, pages 252–265. Lunds Universitets Matematiska Seminarium, Lund, 1952. Reprinted in Œuvres Scientifiques II, [1952b], pp. 48–61.

[Wei72] André Weil. Sur les formules explicites de la théorie des nombres. *Izv. Akad. Nauk SSSR Ser. Mat.*, 36(1):3–18, 1972. English edition: Math. USSR-Izv. 6 (1972), no. 1, 1–17 (article in French); Œuvres III, [1972], 249–264. URL: https://www.mathnet.ru/eng/im2289.

[Wei94] Charles A. Weibel. *An Introduction to Homological Algebra*, volume 38 of *Cambridge Stud. Adv. Math.* Cambridge University Press, 1994.

[Wil00] Andrew Wiles. The Birch and Swinnerton-Dyer conjecture, 2000. Clay Mathematics Institute Millennium Problem description. URL: https://www.claymath.org/wp-content/uploads/2022/05/birchswin.pdf.

[Yos92] Hiroyuki Yoshida. On Hermitian forms attached to zeta functions. In Nobushige Kurokawa and Toshikazu Sunada, editors, *Zeta Functions in Geometry (Tokyo, 1990)*, volume 21 of *Advanced Studies in Pure Mathematics*, pages 281–325. Kinokuniya; Mathematical Society of Japan, Tokyo, 1992. doi:10.2969/aspm/02110281.